

Rough Sheets

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Abstract

Rough sheets are two-parameter analogs of rough paths. In this work the theory of integration over functions of two parameters is extended to cover the case of irregular functions by developing an appropriate notion of rough sheet. The main application is to give a path by path construction of the stochastic integral in the plane and obtain a stratonovich change of variables formula.

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1 Introduction

For processes indexed by one dimensional parameter Lyons' theory of rough paths [11, 12] provide an effective approach to the understanding of relatively complex maps like the stochastic integral or the Itô map which sends the data of an SDE to its solution. These insights were instrumental in proving new results or simplifying the proofs of some important results in stochastic analysis. Just to mention some application: path-wise solutions of SDEs, the support theorem for diffusions, differentiation of the Itô map, the existence of path-wise versions for stochastic currents and the solution of an ODE over Brownian paths driven by a non-adapted vector-field. The theory, being independent from notions like martingales, adaptness, etc... is straightforwardly extended to other, more complex, noises like fractional Brownian motions.

In [8] Gubinelli introduces the notion of *controlled path* in order to abstract the basic structures allowing the integration of rough signals and pointed out the existence of a map (the *sewing map*) which turns out to be at the “core” of the integration process.

In the present paper we study a multi parameter extension to the controlled path theory of [8] and inspired by the structures which one encounters when trying to solve hyperbolic equations driven by irregular signals.

More precisely the example what we have in mind is the stochastic wave equation driven by a fractional Brownian sheet. This equation has been already studied in [19] when the Hurst parameters of the fractional Brownian sheet are strictly bigger than $\frac{1}{2}$ by using the two dimensional Young integration theory. In the cited paper the authors obtain a global existence result in this case in $(1 + 1)$ -dimension.

Our work is a (partially successful) attempt to extend this previous work for the case where the Hurst parameters are less than $\frac{1}{2}$ and for that we will give a definition of a *rough sheet*, which is the basic objects underlying multi-parameter integration suitable to build a theory of path-wise integration over the fractional Brownian sheet. Unfortunately the construction of the rough integral in the multiparameters setting give rise to a very complex algebraic structure which is not well understood at the moment. Due to this difficulties we are not able to construct a full-fledged theory capable of handling solutions to the stochastic wave equation for that reason we will prefer to restrict ourself to the construction of the integral for a restricted family of integrands and to leave the study of the wave equation to further investigations.

More precisely the main result of this paper can be resumed in the following theorem

Theorem 1.1. *Let $\gamma_1, \gamma_2 > 1/3$, then there exist a complete metric space $\mathcal{R}^{\gamma_1, \gamma_2}$ and two continuous application $\mathcal{I}_a : C_b^8(\mathbb{R}) \times \mathcal{R}^{\gamma_1, \gamma_2} \rightarrow \mathcal{CC}_{2,2}^{\gamma_1, \gamma_2}$ (see the equation (29) for the exact definition of $\mathcal{CC}_{2,2}^{\gamma_1, \gamma_2}$), $a = 1, 2$ such that :*

$$\mathcal{I}_1(\varphi, \mathbb{X})_{s_1 s_2; t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) \partial_s \partial_t x_{st} ds dt$$

and

$$\mathcal{I}_2(\varphi, \mathbb{X})_{s_1 s_2; t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) \partial_s x_{st} \partial_t x_{st} ds dt$$

for all $(s_1, s_2, t_1, t_2) \in [0, 1]^4$ and for every smooth sheet x with \mathbb{X} is the rough sheet associated. Moreover if x is a fractional Brownian sheet with Hurst parameter $\gamma_1, \gamma_2 > 1/3$ then it can be enhanced in a rough sheet \mathbb{X} and then we obtain in this case the following Stratonovich formula :

$$\delta\varphi(x) = \mathcal{I}_1(\varphi', \mathbb{X}) + \mathcal{I}_2(\varphi'', \mathbb{X}).$$

We remark that this result is used in [3] to deal with the multi-parametric Skorohod integral defined in [20, 21] and that an explicit formula which give a link between the two notion of integration is obtained.

During the elaboration of this paper two other approaches have been developed to deal with singular parabolic SPDE using rough path ideas. The first one is due to P. Imkeller, N. Perkowski and M. Gubinelli [9] in which the authors use Bony's paraproducts [1] to set up a controlled structure which allow to handle product of distributions in enough generality to be useful to solve PDEs driven by singular signals. A second and more powerful approach has been introduced by M. Hairer in [18] where the author builds a complete non-linear theory for distributions which can be analyzed in terms of a basic set of simple objects called a *model*.

These two extension of rough path theory are very efficient to deal with parabolic SPDE driven by a very irregular multi parameter noise but at the moment while seems possible to adapt these two approaches to handle hyperbolic problems it seems not very clear what would be the final form of a successful theory. In this respect our study and the objects we introduce seems to be needed, maybe in a slightly different form, in order to build a complete theory of hyperbolic rough equations similar to the non-linear stochastic heat equation. Our efforts then can be seen as a first step in this direction.

Plan. This note is structured as follows. In Sec. 2 we recall the basic setup of [8] which allows to embed the theory of rough paths in a theory of integration of “generalized differentials”, called *k-increments*. In Sec. 3 we introduce and study a complex of 2d increments (or *biincrements*) suitable to analyze 2d integrals and show the existence of a 2d Λ -map and of an abstract integration theory (in the sense of convergence of Riemann sums of particular biincrements).

In Sec. 4 we use the theory outlined in Sec. 3 to generalize Young theory of integration to two dimensions. Like in the one-dimensional setting this should be seen as a first (mostly pedagogical) step towards a full theory of rough sheets.

In Sec. 5 we proceed to the dissection of a 2d integral with the purpose of exposing the constituent elements of the would-be rough sheet. All the computations will be done in the smooth setting, emphasizing the algebraic aspects and the respective rôles of the various objects.

In Sec. 5.1 the definition of the rough sheet will be given, and on 5.1.2 a space of sheets *controlled* by a rough sheet, will be introduced and we will show how to obtain an integration theory for them.

In Sec 5.2 we show that for a smooth function φ that $\varphi(x)$ is controlled by x if x is a rough sheet and then obtain some continuity result for the integral constructed in this case using the procedure developed in 5.1.

In Sec 6 we show that the fractional Brownian motion can be enhanced in a rough sheet and then we obtain a Stratonovich change of variable formula in this case.

2 Algebraic integration in one dimension

The integration theory introduced in [8] is based on an algebraic structure, which turns out to be useful for computational purposes, but has also its own interest. Since this setting is quite non-standard,

compared with the one developed in [10], and since it lay at the base of our approach to 2d integrals we will recall briefly here its main features.

2.1 Increments

Let $T > 0$ be an arbitrary positive real number. For any vector space V we introduce a cochain complex $(\mathcal{C}_*(V), \delta)$ as follows. a k -*increment* with values in V is a function $g : [0, T]^k \rightarrow V$, such that $g_{t_1 \dots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i = 1, \dots, k$. Denote with $\mathcal{C}_k(V)$ the corresponding set. On k -increments, define a the following coboundary operator δ :

$$\delta : \mathcal{C}_k(V) \rightarrow \mathcal{C}_{k+1}(V) \quad (\delta g)_{t_1 \dots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i+1} g_{t_1 \dots \hat{t}_i \dots t_{k+1}} \quad (1)$$

where \hat{t}_i means that this particular argument is omitted. It is easy to verify that $\delta\delta = 0$. We will denote $\mathcal{Z}\mathcal{C}_k(V) = \mathcal{C}_k(V) \cap \text{Ker}\delta$ and $\mathcal{B}\mathcal{C}_k(V) := \mathcal{C}_k(V) \cap \text{Im}\delta$, respectively the spaces of k -*cocycles* and of k -*coboundaries* following standard conventions of homological algebra. We will write \mathcal{C}_k when the underlying vector space is \mathbb{R} .

Some simple examples of actions of δ are obtained by letting $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then, for any $t, u, s \in [0, T]$, we have

$$(\delta g)_{ts} = g_t - g_s, \quad \text{and} \quad (\delta h)_{tus} = h_{ts} - h_{tu} - h_{us}. \quad (2)$$

The complex $(\mathcal{C}_*(V), \delta)$ is *acyclic*, i.e. $\mathcal{Z}\mathcal{C}_{k+1}(V) = \mathcal{B}\mathcal{C}_k(V)$ for any $k \geq 0$ or otherwise stated, the sequence

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}_1(V) \xrightarrow{\delta} \mathcal{C}_2(V) \xrightarrow{\delta} \mathcal{C}_3(V) \xrightarrow{\delta} \mathcal{C}_4(V) \rightarrow \dots \quad (3)$$

is exact.

This exactness implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_1(V)}$: it measures how much a given 1-increment is far from being an *exact* increment of a function (i.e. a finite difference).

For our discussion only k -increments with $k \leq 3$ will be relevant. When V is a Banach space with norm $|\cdot|$ we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu \equiv \sup_{s, t \in [0, T]} \frac{|f_{st}|}{|t - s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V); \|f\|_\mu < \infty\}.$$

In the same way for $h \in \mathcal{C}_3(V)$ set

$$\begin{aligned} \|h\|_{\gamma, \rho} &= \sup_{s, u, t \in [0, T]} \frac{|h_{sut}|}{|u - s|^\gamma |t - u|^\rho} \\ \|h\|_\mu &\equiv \inf \left\{ \sum_i \|h_i\|_{\rho_i, \mu - \rho_i}; h = \sum_i h_i, 0 < \rho_i < \mu \right\}, \end{aligned} \quad (4)$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, \mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V); \|h\|_\mu < \infty\}.$$

Let $\mathcal{C}_3^{1+}(V) = \cup_{\mu>1} \mathcal{C}_3^\mu(V)$. Analogous meaning should be given to $\mathcal{Z}\mathcal{C}_2^\mu(V), \dots$

From now on V will be a generic Banach space.

The following proposition is a basic result which is at the core of our approach to path-wise integration:

Proposition 2.1 (The Λ -map). *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \rightarrow \mathcal{C}_2^{1+}(V)$ such that $\delta\Lambda = 1_{\mathcal{Z}\mathcal{C}_3(V)}$. Furthermore, for any $\mu > 1$ this map is continuous from $\mathcal{Z}\mathcal{C}_2^\mu(V)$ to $\mathcal{C}_1^\mu(V)$ and we have*

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2} \|h\|_\mu, \quad h \in \mathcal{Z}\mathcal{C}_2^{1+}(V) \quad (5)$$

Proof. See [8]. □

Let $\mathcal{R}^\mu(V) = \{g \in \mathcal{C}_2(V) : \delta g \in \mathcal{C}_3^\mu(V)\}$. When $\mu > 1$ this is the subspace of 1-increments whose coboundary is small enough to be in the domain of Λ . $\mathcal{R}^{1+}(V)$ is defined as the union of all $\mathcal{R}^\mu(V)$ for $\mu > 1$.

An immediate implication of Prop. 2.1 is the following algorithm for a canonical decomposition of the elements of $\mathcal{R}^{1+}(V)$:

Corollary 2.2. *Take an element $g \in \mathcal{R}^\mu(V)$ for $\mu > 1$. Then g can be decomposed in a unique way as $g = \delta f + \Lambda \delta g$, where $f \in \mathcal{C}_1(V)$. If moreover $g \in \mathcal{C}_2^{1+}(V)$ then $f = 0$ and $g = \Lambda \delta g$: the coboundary δ is invertible (as a linear map) in $\mathcal{C}_1^{1+}(V)$.*

Proof. By assumption there are no ambiguity to define $\Lambda \delta g$ and by definition we have that $\delta(g - \Lambda \delta g) = 0$ and then there exist $f \in \mathcal{C}_1(V)$ such that $\delta f = g - \Lambda \delta g$. In the case when $g \in \mathcal{C}_2^{1+}$ we see that f is a γ -Hölder function for some $\gamma > 1$ and then $f_t = f_0$ which gives our result. □

At this point the relation of the structure we introduced with the problem of integration of irregular functions can be still quite obscure to the non-initiated reader. However something interesting is already going on and the previous corollary has a very nice consequence which is the subject of the next corollary.

Corollary 2.3 (Integration of small increments). *For any 1-increment $g \in \mathcal{C}_2(V)$, such that $\delta g \in \mathcal{C}_1^{1+}$ set $\delta f = (1 - \Lambda \delta)g$, then*

$$(\delta f)_{ts} = \lim_{|\Pi_{ts}| \rightarrow 0} \sum_{i=0}^n g_{t_i t_{i+1}}$$

where the limit is over all partitions $\Pi_{ts} = \{t_0 = t, \dots, t_n = s\}$ of $[t, s]$ as the size of the partition goes to zero. The 1-increment δf is the indefinite integral of the 1-increment g .

Even if the result is already in [8] we would like to repeat the proof since it is quite illuminating and it will be source of inspiration when proving a similar statement in the 2d setting.

Proof. Just consider the equation $g = \delta f + \Lambda \delta g$ and write

$$S_\Pi = \sum_{i=0}^n g_{t_i t_{i+1}} = \sum_{i=0}^n (\delta f)_{t_i t_{i+1}} + \sum_{i=0}^n (\Lambda \delta g)_{t_i t_{i+1}} = (\delta f)_{ts} + \sum_{i=0}^n (\Lambda \delta g)_{t_i t_{i+1}}$$

and observe that, due to the fact that $\Lambda \delta g \in \mathcal{C}_1^{1+}(V)$ the last sum converges to zero. □

2.1.1 Computations in \mathcal{C}_*

If V is an associative algebra the complex (\mathcal{C}_*, δ) is an (associative, non-commutative) graded algebra once endowed with the following product: for $g \in \mathcal{C}_n(V)$ and $h \in \mathcal{C}_m(V)$ let $gh \in \mathcal{C}_{n+m-1}(V)$ the element defined by

$$(gh)_{t_1, \dots, t_{m+n+1}} = g_{t_1, \dots, t_n} h_{t_{n+1}, \dots, t_{m+n+1}}, \quad t_1, \dots, t_{m+n+1} \in [0, T]. \quad (6)$$

The coboundary δ act as a graded derivation with respect to the algebra structure. In particular we have the following useful properties.

Proposition 2.4. *The following differentiation rules hold:*

1. Let g, h be two elements of $\mathcal{C}_1(V)$. Then

$$\delta(gh) = \delta g h + g \delta h. \quad (7)$$

2. Let $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then

$$\delta(gh) = -\delta g h + g \delta h, \quad \delta(hg) = \delta h g + h \delta g.$$

Proof. We will just prove (7), the other relations being equally trivial: if $g, h \in \mathcal{C}_1(V)$, then

$$[\delta(gh)]_{ts} = g_t h_t - g_s h_s = g_t (h_t - h_s) + (g_t - g_s) h_s = g_t (\delta h)_{ts} + (\delta h)_{ts} g_s,$$

which proves our claim. □

The iterated integrals of smooth functions on $[0, T]$ are of course particular cases of elements of \mathcal{C} which will be of interest for us. Let us recall some basic rules for these objects: consider $f, g \in \mathcal{C}_1^\infty$, where \mathcal{C}_1^∞ is the set of smooth functions from $[0, T]$ to \mathbb{R} . Then the integral $\int f dg$, that we will denote by $\mathcal{J}(fdg)$, can be considered as an element of \mathcal{C}_2^∞ . That is, for $s, t \in [0, T]$, we set

$$\mathcal{J}_{ts}(fdg) = \left(\int f dg \right)_{ts} = \int_s^t f_u dg_u.$$

The multiple integrals can also be defined in the following way: given a smooth element $h \in \mathcal{C}_2^\infty$ and $s, t \in [0, T]$, we set

$$\mathcal{J}_{ts}(hdg) \equiv \left(\int hdg \right)_{ts} = \int_s^t h_{us} dg_u.$$

In particular, the double integral $\mathcal{J}_{ts}(f^1 df^2 df^3)$ is defined, for $f^1, f^2, f^3 \in \mathcal{C}_0^\infty$, as

$$\mathcal{J}_{ts}(f^1 df^2 df^3) = \left(\int f^1 df^2 df^3 \right)_{ts} = \int_s^t \mathcal{J}_{us}(f^1 df^2) df_u^3.$$

Now, suppose that the n th order iterated integral of $f^1 df^2 \dots df^n$, still denoted by the expression $\mathcal{J}(f^1 df^2 \dots df^n)$, has been defined for $f^1, f^2, \dots, f^n \in \mathcal{C}_1^\infty$. Then, if $f^{n+1} \in \mathcal{C}_1^\infty$, we set

$$\mathcal{J}_{ts}(f^1 df^2 \dots df^n df^{n+1}) = \int_s^t \mathcal{J}_{us}(f^1 df^2 \dots df^n) df_u^{n+1}, \quad (8)$$

which defines the iterated integrals of smooth functions recursively. Observe that a n th order integral $\mathcal{J}(df^1 df^2 \cdots df^n)$ could be defined along the same lines.

The following relations between multiple integrals and the operator δ will also be useful in the remainder of the paper:

Proposition 2.5. *Let f, g be two elements of \mathcal{C}_1^∞ . Then, recalling the convention (6); it holds that*

$$\delta f = \mathcal{J}(df), \quad \delta(\mathcal{J}(f dg)) = 0, \quad \delta(\mathcal{J}(dg df)) = (\delta g)(\delta f) = \mathcal{J}(dg) \mathcal{J}(df),$$

and, in general,

$$\delta(\mathcal{J}(df^n \cdots df^1)) = \sum_{i=1}^{n-1} \mathcal{J}(df^n \cdots df^{i+1}) \mathcal{J}(df^i \cdots df^1). \quad (9)$$

Proof. Here again, the proof is elementary, and we will just show the third of these relations: we have, for $s, t \in [0, T]$,

$$\mathcal{J}_{ts}(dg df) = \int_s^t dg_u(f_u - f_s) = \int_s^t dg_u f_u - K_{ts},$$

with $K_{ts} = (g_t - g_s)f_s$. The first term of the right hand side is easily seen to be in $\ker \delta_{\Sigma_2^*}$. Thus

$$\delta(\mathcal{J}(dg df))_{tus} = (\delta K)_{tus} = [g_t - g_u][f_u - f_s],$$

which gives the announced result. □

2.1.2 Dissection of an integral

To grasp the algorithm underling the rough-path approach to integrals over irregular functions we will exercise ourselves on the deconstruction of a “classic” integral.

With the notations of Sec. 2.1.1 in mind, we will split the integral $\int \varphi(x) dx = \mathcal{J}(\varphi(x) dx)$ for a smooth function $x \in \mathcal{C}_1$ into “more elementary” components. This decomposition suggest the right structure for the 1d rough paths. A similar exercise for 2d integrals will be very important in understanding the correct structure of the rough sheets.

The first idea one can have in mind in order to analyze $\mathcal{J}(\varphi(x) dx)$ is to perform an expansion around the increment dx . By Taylor expansion we have

$$\mathcal{J}(\varphi(x) dx) = \varphi(x) \mathcal{J}(dx) + \mathcal{J}(d\varphi(x) dx). \quad (10)$$

The first term in the r.h.s will be considered “elementary” and not elaborated further. Note that it is defined independently of the regularity of x since $\varphi(x) \mathcal{J}(dx) = \varphi(x) \delta x$.

Moreover, as a 1-increment it is easy to see that the second term in the r.h.s. of eq. (10) is smaller than the first but more problematic and we proceed to its *dissection* by the application of δ : invoking Proposition 2.5, we get that

$$\delta(\mathcal{J}(d\varphi(x) dx)) = \delta(\varphi(x)) \delta x, \quad (11)$$

Now the r.h.s. is well defined independently of the regularity of x since

$$[\delta(\varphi(x)) \delta x]_{tus} = (\varphi(x_t) - \varphi(x_u))(x_u - x_s).$$

Since x is smooth and assuming that φ is differentiable, then $\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x) \in \mathcal{C}_1^{1+}$ (since actually it belongs to \mathcal{C}_1^2 by easy bounds on the iterated integral). Then as a consequence of Corollary 2.2 we have that

$$\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x) = \Lambda\delta[\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x)] = -\Lambda[\delta(\varphi(x))\delta x]. \quad (12)$$

and, as a result, the following expression for the original integral holds

$$\mathcal{J}(\varphi(x)\mathrm{d}x) = \varphi(x)\delta x - \Lambda(\delta(\varphi(x))\delta x) = (1 - \Lambda\delta)[\varphi(x)\delta x]. \quad (13)$$

Eq. (13) shows that the ordinary integral on the l.h.s. is equivalent to an expression in the r.h.s. which does not depend any more on any differentiability assumptions on x , indeed the r.h.s. makes sense, for example, when $\varphi \in \text{Lip}$ and $x \in \mathcal{C}_0^\gamma$ for any $\gamma > 1/2$: the only thing we have to check is that $\delta(\varphi(x))\delta x \in \mathcal{C}_2^{1+}$ but under these assumptions we have

$$|(\varphi(x_t) - \varphi(x_u))(x_u - x_s)| \leq L_\varphi \|x\|_\gamma^2 \|\varphi(x)\|_\gamma |t - u|^\gamma |u - s|^\gamma$$

where $\|\cdot\|_\gamma$ is the ordinary γ -Hölder norm on functions and L_φ is the Lipschitz norm of φ . In this case we can *define* the integral in the l.h.s. as being equivalent to the well-defined r.h.s. and this new integral is essentially the integral introduced by Young in [22]. What is really relevant to our discussion is to note that the integral can, in this case, be completely recovered from the 1-increment $\varphi(x)\delta x$.

However, the procedure can be continued further on by the next step in the Taylor expansion of the integral (10), which reads, for $s, t \in [0, T]$,

$$\int_s^t [\mathrm{d}_u \varphi(x_u)] = \int_s^t \varphi'(x_u) \mathrm{d}x_u = \varphi'(x_s)[x_t - x_s] + \int_s^t \left(\int_s^u \varphi'(x_v) \mathrm{d}x_v \right) \mathrm{d}x_u,$$

or according to the notations of Section 2.1.1,

$$\delta\varphi(x) = \mathcal{J}(\mathrm{d}\varphi(x)) = \mathcal{J}(\varphi'(x)\mathrm{d}x) = \varphi'(x)\mathcal{J}(\mathrm{d}x) + \mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x). \quad (14)$$

Injecting this equality in equation (10), thanks to (8), we obtain

$$\mathcal{J}(\varphi(x)\mathrm{d}x) = \varphi(x)\mathcal{J}(\mathrm{d}x) + \varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x) + \mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x\mathrm{d}x). \quad (15)$$

In the two first terms in the r.h.s of eq. (15) the function $\varphi(x)$ has “pop out” from the integral, so we consider them elementary (in a sense we will discuss below). Again, the last term in the r.h.s. can be seen to belong to \mathcal{C}_1^{1+} (since actually, in this smooth setting, it belongs to \mathcal{C}_1^3). Then in analogy with eq. (12) we can represent it in terms of its image under δ as

$$\mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x\mathrm{d}x) = -\Lambda\delta[\mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x\mathrm{d}x)] = -\Lambda[\mathcal{J}(\mathrm{d}\varphi'(x))\mathcal{J}(\mathrm{d}x\mathrm{d}x) + \mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x)\mathcal{J}(\mathrm{d}x)] \quad (16)$$

where we acted with δ upon the triple iterated integral according to Prop. 2.5. Concerning the argument of Λ in this last equation, we note the following two facts: $\mathcal{J}(\mathrm{d}\varphi'(x)) = \delta\varphi'(x)$ while the double iterated integral $\mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x)$ appears in the Taylor expansion for $\delta\varphi(x)$:

$$\delta\varphi(x) = \mathcal{J}(\mathrm{d}\varphi(x)) = \mathcal{J}(\varphi'(x)\mathrm{d}x) = \varphi'(x)\mathcal{J}(\mathrm{d}x) + \mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x)$$

so

$$\mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x) = \delta\varphi(x) - \varphi'(x)\delta x. \quad (17)$$

Then we can rewrite eq. (16) as

$$\mathcal{J}(\mathrm{d}\varphi'(x)\mathrm{d}x\mathrm{d}x) = -\Lambda[\delta\varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x) + (\delta\varphi(x) - \varphi'(x)\delta x)\mathcal{J}(\mathrm{d}x)] \quad (18)$$

and finally we have obtained another expression for the integral $\mathcal{J}(\varphi(x)\mathrm{d}x)$:

$$\begin{aligned} \mathcal{J}(\varphi(x)\mathrm{d}x) &= \varphi(x)\delta x + \varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x) \\ &\quad - \Lambda[\delta\varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x) + (\delta\varphi(x) - \varphi'(x)\delta x)\mathcal{J}(\mathrm{d}x)] \\ &= (1 - \Lambda\delta)[\varphi(x)\delta x + \varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x)] \end{aligned} \quad (19)$$

where to go from the first equation to the second we used the algebraic relation

$$\delta\mathcal{J}(\mathrm{d}x\mathrm{d}x) = \mathcal{J}(\mathrm{d}x)\mathcal{J}(\mathrm{d}x). \quad (20)$$

Up to this point all we got are another equivalent expression for the classic integral in the r.h.s. of eq. (18).

It is a very remarkable basic result of rough path theory that the r.h.s. of eq. (18) makes sense for paths x which are very irregular like the sample paths of Brownian motion (which a.s. are not Hölder continuous for any index greater than 1/2), once we have at our disposal also a 1-increment $\mathcal{J}(\mathrm{d}x\mathrm{d}x)$ which is sufficiently small and satisfy eq. (20). Heuristically, in this situation the formula says that the 1-increment $\varphi(x)\delta x$ can be “corrected” or “renormalized” by adding the correction $\varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x)$ so that it becomes integrable (in the sense of Corollary 2.3).

In the cases where this correction belongs to \mathcal{C}_1^{1+} we have $(1 - \Lambda\delta)[\varphi'(x)\mathcal{J}(\mathrm{d}x\mathrm{d}x)] = 0$ so eq. (18) becomes again eq. (12) and we reobtain the Young integral.

It is worth noticing at that point that the integral, as *defined* by eq. (18), has now to be understood as an integral over the (step-2) rough path $(x, \mathcal{J}(\mathrm{d}x\mathrm{d}x))$ [8] and it coincide with the notion of integral over a rough path given by Lyons in [12].

This algorithm has an obvious extension to higher orders if we assume that a reasonable definition of the iterated integrals $\mathcal{J}(\mathrm{d}x\mathrm{d}x \cdots \mathrm{d}x)$ can be given. To proceed further however we need the notion of *geometric* rough path (for more details on this notion see [12]) which must be exploited crucially to show that some terms are small and belongs to the domain of Λ .

Note that we have worked in the scalar setting (i.e. all the object we considered are real-valued). Willing to add some notational burden it is easy to see that this section has an equivalent formulation in the vector case (when x takes values on \mathbb{R}^n and φ is a (smooth) differential form on \mathbb{R}^n). Indeed all the theory is interesting and useful especially in the vector case. This explain the fact that we do not considered techniques like the Doss-Sussmann approach to define one dimensional integrals since they are essentially limited to the scalar setting (where every reasonable differential form φ is exact) and do not have a vectorial counterpart.

3 The increment complex in two dimensions

In this paper we are interested in particular two-dimensional integrals which can take two basic forms which in general are not equivalent. If $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are regular enough we can define the two dimensional integral of f wrt. g as

$$\iint_{(s_1, t_1)}^{(s_2, t_2)} f \mathrm{d}g := \int_{s_1}^{s_2} \mathrm{d}s \int_{t_1}^{t_2} \mathrm{d}t f(s, t) \partial_1 \partial_2 g(s, t) \quad (21)$$

where ∂_1 and ∂_2 are the partial derivatives wrt. the first and the second coordinate, respectively. Another possible and nonequivalent basic integral in two-dimension is given, for a triple of functions f, g, h , by

$$\iint_{(s_1, t_1)}^{(s_2, t_2)} f d_1 g d_2 h := \iint_{(s_1, t_1)}^{(s_2, t_2)} f(s, t) \partial_1 g(s, t) \partial_2 h(s, t) ds dt$$

Then

$$\iint_{(s_1, t_1)}^{(s_2, t_2)} dg =: (\delta g)(s_1, t_1, s_2, t_2) \quad (22)$$

defines the coboundary map δ for functions of two parameters:

$$\begin{aligned} (\delta g)(s_1, t_1, s_2, t_2) &= g(s_2, t_2) - g(s_1, t_2) - g(s_2, t_1) + g(s_1, t_1) \\ &= [g(s_2, t_2) - g(s_1, t_2)] - [g(s_2, t_1) - g(s_1, t_1)] \end{aligned} \quad (23)$$

which is just the composition of two finite-difference operator in the two directions. These integrals are to be considered as continuous functions of two points (s_1, t_1) and (s_2, t_2) on the plane which vanishes whenever $s_1 = s_2$ or $t_1 = t_2$. This preliminary observation leads to the following general construction for a 2d cochain complex suitable for the analysis of these two-parameter integrals.

Fix a positive real T and let $\mathcal{CC}_{k,l}(V)$ the space of continuous functions from $[0, T]^k \times [0, T]^l \rightarrow V$, V some vector space such that

$$g_{(s_1, \dots, s_k)(t_1, \dots, t_l)} = 0$$

whenever $s_i = s_{i+1}$ or $t_i = t_{i+1}$. We will write $\mathcal{CC}_{k,l} = \mathcal{CC}_{k,l}(\mathbb{R})$.

For $\mathcal{CC}_k(V) = \mathcal{CC}_{k,k}(V)$ we will use the natural identification with the space of continuous functions from $([0, T]^2)^k \rightarrow V$. These will play the role of 2d k -increments: they are functions of k points in the square $[0, T]^2$ such that they become zero whenever two contiguous arguments have one coordinate in common.

Note that $\mathcal{CC}_{k,l} = \mathcal{C}_k \otimes \mathcal{C}_l$ and in general

$$\mathcal{CC}_{k,l}(V) = \mathcal{C}_k \otimes \mathcal{C}_l \otimes V \quad (24)$$

We will call the elements of $\mathcal{CC}_{k,l}(V)$ (k, l) -biincrements and the elements of $\mathcal{CC}_k(V)$ k -biincrements. Moreover we introduce one-dimensional coboundaries δ_1, δ_2 which acts as described in Sec. 2 on the biincrements view as functions of the first set, or of the second set of arguments, i.e. they acts on the first or second \mathcal{C}_* factor according to factorization of eq. (24). To be concrete

$$\delta_1 : \mathcal{CC}_{k,l}(V) \rightarrow \mathcal{CC}_{k+1,l}(V)$$

$$\delta_2 : \mathcal{CC}_{k,l}(V) \rightarrow \mathcal{CC}_{k,l+1}(V)$$

and for example, if $g \in \mathcal{CC}_{k,l}(V)$ then

$$(\delta_1 g)_{(s_1, \dots, s_{k+1}), (t_1, \dots, t_l)} = \sum_{i=1}^{k+1} (-1)^{i+1} g_{(s_1, \dots, \hat{s}_i, \dots, s_{k+1}), (t_1, \dots, t_l)}$$

where, as usual, the notation \hat{s}_i means that the corresponding argument is omitted. It is easy to see that δ_1 and δ_2 commute and that

$$\delta = \delta_1 \delta_2 : \mathcal{CC}_k(V) \rightarrow \mathcal{CC}_{k+1}(V)$$

is a coboundary, i.e. satisfy the equation $\delta\delta = 0$. Moreover, if $g \in \mathcal{CC}_{k,l}(V)$ we have

$$(\delta g)_{(s_1, \dots, s_{k+1}), (t_1, \dots, t_{l+1})} = \sum_{i=1}^{k+1} \sum_{j=1}^{l+1} (-1)^{i+j} g_{(s_1, \dots, \hat{s}_i, \dots, s_{k+1}), (t_1, \dots, \hat{t}_j, \dots, t_{l+1})}$$

Then $(\mathcal{CC}_*(V), \delta)$ is a cochain complex. It will be important to note that its cohomology is not trivial and that it will play a role in our subsequent results.

3.1 Cohomology of (\mathcal{CC}_*, δ)

The complex $(\mathcal{CC}_*(V), \delta)$ is the diagonal of the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{CC}_{1,1}(V) & \xrightarrow{\delta_1} & \mathcal{CC}_{2,1}(V) & \xrightarrow{\delta_1} & \mathcal{CC}_{3,1}(V) & \xrightarrow{\delta_1} & \dots \\ \delta_2 \downarrow & & \delta_2 \downarrow & & \delta_2 \downarrow & & \\ \mathcal{CC}_{1,2}(V) & \xrightarrow{\delta_1} & \mathcal{CC}_{2,2}(V) & \xrightarrow{\delta_1} & \mathcal{CC}_{3,2}(V) & \xrightarrow{\delta_1} & \dots \\ \delta_2 \downarrow & & \delta_2 \downarrow & & \delta_2 \downarrow & & \\ \mathcal{CC}_{1,3}(V) & \xrightarrow{\delta_1} & \mathcal{CC}_{2,3}(V) & \xrightarrow{\delta_1} & \mathcal{CC}_{3,3}(V) & \xrightarrow{\delta_1} & \dots \\ \delta_2 \downarrow & & \delta_2 \downarrow & & \delta_2 \downarrow & & \end{array} \quad (25)$$

We are mainly interested in the first cohomology group

$$H_1(\mathcal{CC}, \delta) = \frac{\mathcal{ZCC}_1(V)}{\mathcal{BCC}_1(V)}$$

where as before we denote $\mathcal{ZCC}_k(V) = \text{Ker} \delta|_{\mathcal{CC}_k(V)}$ and $\mathcal{BCC}_k(V) = \text{Im} \delta|_{\mathcal{CC}_{k-1}(V)}$, the spaces of k -bicycles and $k-1$ -bicoboundaries, respectively.

To compute the cohomology consider applications $\sigma_1 : \mathcal{CC}_{k,l}(V) \rightarrow \mathcal{CC}_{k-1,l}(V)$ (for $k \geq 1$) and $\sigma_2 : \mathcal{CC}_{k,l}(V) \rightarrow \mathcal{CC}_{k,l-1}(V)$ (for $l \geq 1$) which fix the first argument on each direction to the (arbitrary) value 0. For example:

$$(\sigma_1 g)_{(s_1, \dots, s_{k-1}), (t_1, \dots, t_l)} = g_{(0, s_1, \dots, s_{k-1}), (t_1, \dots, t_l)}$$

and a similar equation for σ_2 . Then we have the homotopy formulas

$$\sigma_i \delta_i - \delta_i \sigma_i = 1, \quad \text{for } i=1,2$$

which are at the origin of the exactness of the one-dimensional complexes forming the rows and the columns of the diagram (25). Let $k \geq 1$. Take $a \in \mathcal{ZCC}_k(V)$ and let

$$b = a - \sigma_1 \delta_1 a - \sigma_2 \delta_2 a$$

and, using the homotopy formulas, verify that

$$\delta_1 b = \delta_1 a - \delta_1 a - \delta_1 \sigma_2 \delta_2 a = -\sigma_2 \delta a = 0$$

since δ_1 commutes with σ_2 . Similarly $\delta_2 b = 0$. Then $b \in \text{Ker} \delta_1 \cap \text{Ker} \delta_2$ which means that we can write

$$b = \delta_1 s_1 b = \delta_1 \sigma_1 \delta_2 \sigma_2 b$$

but since operators with different indexes commutes we can always rewrite this as

$$b = \delta_1 \delta_2 \sigma_1 \sigma_2 b = \delta \sigma_1 \sigma_2 b$$

so that $b \in \mathcal{BCC}_k(V)$. Next, note that $\sigma_1 \delta_1 a \in \text{Ker} \delta_2$, since $\delta_2 \sigma_1 \delta_1 a = \sigma_1 \delta a = 0$ so $\sigma_1 \delta_1 a = \delta_2 \sigma_2 \sigma_1 \delta_1 a$. Then for any $a \in \mathcal{ZCC}_k(V)$ we have the decomposition

$$a = \delta q + \delta_1 \sigma \delta_2 a + \delta_2 \sigma \delta_1 a \quad (26)$$

for some $q \in \mathcal{CC}_{k-1}(V)$ where we let $\sigma = \sigma_1 \sigma_2 = \sigma_2 \sigma_1$.

3.2 Computations in $\mathcal{CC}_{*,*}$

For $a \in \mathcal{CC}_{n,m}$, $b \in \mathcal{CC}_{k,l}$ we can define the (noncommutative, associative) product $ab \in \mathcal{CC}_{n+k-1,m+l-1}$ as

$$ab_{(s_1, \dots, s_{n+k-1})(t_1, \dots, t_{m+l-1})} = a_{(s_1, \dots, s_n)(t_1, \dots, t_m)} b_{(s_{n+1}, \dots, s_{n+k-1})(t_{m+1}, \dots, t_{m+l-1})}.$$

For example for $a \in \mathcal{CC}_{2,1}$, $b \in \mathcal{CC}_{1,2}$ we have $(ab)_{(s_1, s_2)(t_1, t_2)} = a_{(s_1, s_2)t_1} b_{s_2(t_1, t_2)}$. This definition is suited to work well with the action of δ_1, δ_2 , for example we have that, if $a, b \in \mathcal{CC}_{1,*}$:

$$\delta_1(a\delta_1 b) = \delta_1 a \delta_1 b, \quad \delta_1(\delta_1 b a) = -\delta_1 b \delta_1 a$$

If $a, b \in \mathcal{CC}_1$:

$$\begin{aligned} \delta_2(a\delta_1 b) &= \delta_2 a \delta_1 b + a \delta b, \\ \delta(\delta_1 a \delta_2 b) &= \delta_1((\delta_2 \delta_1 a) \delta_2 b) + \delta_1 a (\delta_2 \delta_2 b) = \delta_1(\delta a \delta_2 b) = -\delta a \delta b \end{aligned}$$

and

$$\delta(\delta_2 b \delta_1 a) = -\delta b \delta a, \quad \delta(a \delta b) = \delta(\delta a b) = \delta a \delta b$$

as can be easily checked by a direct computation. In the two parameters setting we will improve our algebraic structure by adding a new type of product for $f \in \mathcal{CC}_{m,n}$ and $g \in \mathcal{CC}_{m,l}$ we define $f \circ_1 g \in \mathcal{CC}_{m,n+l}$ by

$$f \circ_1 g_{(s_1 s_2 \dots s_m)(t_1 t_2 \dots t_{n+l-1})} = f_{(s_1 s_2 \dots s_m)(t_1 t_2 \dots t_n)} g_{(s_{n+1} s_{n+2} \dots s_m)(t_{n+1} \dots t_{n+l-1})}$$

and an analogue definition in the second direction .

For a two-dimensional quantity like the basic integral in eq.(21) we can write down the following relation

$$\begin{aligned} & \iint_{(s_1, t_1)}^{(s_2, t_2)} \left[\iint_{(s_1, t_1)}^{(s, t)} df(u, v) \right] dg(s, t) \\ &= \iint_{(s_1, t_1)}^{(s_2, t_2)} [f(s, t) - f(s_1, t) - f(s, t_1) + f(s_1, t_1)] dg(s, t) \\ &= \iint_{(s_1, t_1)}^{(s_2, t_2)} f(s, t) dg(s, t) + f(s_1, t_1) (\delta g)(s_1, t_1, s_2, t_2) \\ &\quad - \int_{t_1}^{t_2} f(s_1, t) d_2 [g(s_2, t) - g(s_1, t)] \\ &\quad - \int_{s_1}^{s_2} f(s, t_1) d_1 [g(s, t_2) - g(s, t_1)] \end{aligned} \quad (27)$$

which will play the same role as eq. (10) in the one-dimensional setting.

As an example of the formalism set up up to now we can consider the decomposition of eq. (27) for the (two-dimensional) iterated integral $\iint df dg$ of two smooth elements $f, g \in \mathcal{C}\mathcal{C}_1$. In compact notation it reads:

$$\iint f dg = -f\delta g + \int f d_1\delta_2 g + \int f d_2\delta_1 g + \iint df dg$$

where we understand the integrals as functions of both the extremes of integration. Note that

$$\delta \iint df dg = \delta(f\delta g) = \delta f\delta g$$

so

$$\iint f dg - \int f d_1\delta_2 g - \int f d_2\delta_1 g \in \ker \delta. \quad (28)$$

Actually the three factors in eq. (28) corresponds exactly to the decomposition (26) since

$$\iint f dg = \delta \iint_{*,*} f dg, \quad \int f d_1\delta_2 g = \delta_1 \int_* f d_1\delta_2 g, \quad \int f d_2\delta_1 g = \delta_2 \int_* f d_2\delta_1 g$$

where the star in the integral sign denote that the lower integration point has been fixed arbitrarily (e.g. to $0 \in [0, T]$).

Another relevant remark is to note that the antisymmetric element $\omega^a = \delta_1 f\delta_2 g - \delta_2 f\delta_1 g$ satisfy $\delta\omega^a = 0$. Indeed

$$\delta_1\omega^a = -\delta_1 f\delta g - \delta f\delta_1 g, \quad \delta\omega^a = -\delta f\delta g + \delta f\delta g = 0$$

according to the rules established above. For the symmetric counterpart $\omega^s = \delta_1 f\delta_2 g + \delta_2 f\delta_1 g$ we have

$$\delta_1\omega^s = -\delta_1 f\delta g + \delta f\delta_1 g, \quad \delta\omega^s = -\delta f\delta g - \delta f\delta g = -2\delta f\delta g$$

3.3 Splitting and other operations

Each of the vector spaces $\mathcal{C}\mathcal{C}_{k,m}(V)$ is naturally isomorphic to either $\mathcal{C}_k(\mathcal{C}_m(V))$ or to $\mathcal{C}_m(\mathcal{C}_k(V))$: consider each k, m -biincrement either as a k -increment in the first direction with values in m -increments in the other direction or vice-versa. The multiplication in $\mathcal{C}\mathcal{C}_{*,*}$ is compatible with these isomorphism.

In what follows it will be useful to introduce a one-dimensional *splitting* map S which sends products $ab \in \mathcal{C}_2(V)$ for $a \in \mathcal{C}_1(V)$ and $b \in \mathcal{C}_1(V)$ to the elementary tensor $S(ab) = a \otimes b \in \mathcal{C}_1(V) \overset{V}{\otimes} \mathcal{C}_1(V)$ where the tensor product is over the algebra V . The map is extended by linearity to the subspace \mathcal{M}_2 of \mathcal{C}_2 generated by the linear combinations of products of two elements of \mathcal{C}_1 . Elements of $\mathcal{C}_1(V) \overset{V}{\otimes} \mathcal{C}_1(V)$ are just functions $(t, u, v, s) \mapsto c_{tuv s}$ of four arguments which are 1-increments in the couple (t, u) and in the couple (v, s) but which may be non-zero for $u = v$. The multiplication map $\mu : \mathcal{C}_1(V) \overset{V}{\otimes} \mathcal{C}_1(V) \rightarrow \mathcal{C}_2(V)$ just sends each c to the 2-increment $(t, u, s) \mapsto a_{tuv s}$ and is the inverse of S : $\mu \circ S(a) = a$ for any $a \in \mathcal{M}_2$.

We will denote $S_1 : \mathcal{C}\mathcal{C}_{2,k}(V) \rightarrow \mathcal{C}_1(\mathcal{C}_k(V)) \overset{\mathcal{C}_k(V)}{\otimes}_1 \mathcal{C}_1(\mathcal{C}_k(V))$ and $S_2 : \mathcal{C}\mathcal{C}_{k,2}(V) \rightarrow \mathcal{C}_1(\mathcal{C}_k(V)) \overset{\mathcal{C}_k(V)}{\otimes}_2 \mathcal{C}_1(\mathcal{C}_k(V))$ the splitting maps according to the first or the second direction. These are understood according to the above isomorphism $\mathcal{C}_{k,m}(V) \simeq \mathcal{C}_k(\mathcal{C}_m(V)) \simeq \mathcal{C}_m(\mathcal{C}_k(V))$ and the index 1, 2 on the tensor product remember in which of the two directions the splitting has taken place.

3.4 Abstract integration in \mathcal{CC}_*

From now on we assume that V is a Banach space with norm $|\cdot|$. When they appears tensor product will be understood according to the projective topology.

Let us introduce the following norms, for any $g \in \mathcal{CC}_2(V)$

$$\|g\|_{z_1, z_2} := \sup_{s, t \in [0, T]^2} \frac{|g(s_1, t_1)(s_2, t_2)|}{|s_1 - t_1|^{z_1} |s_2 - t_2|^{z_2}} \quad (29)$$

and for $h \in \mathcal{CC}_3(V)$

$$\|h\|_{\gamma_1, \gamma_2, \rho_1, \rho_2} := \sup_{s, u, t \in [0, T]^2} \frac{|h(s_1, u_1, t_1)(s_2, u_2, t_2)|}{\prod_{i=1,2} |s_i - u_i|^{\gamma_i} |u_i - t_i|^{\rho_i}}$$

and

$$\|h\|_{z_1, z_2} := \inf \left\{ \sum_i \|h_i\|_{\gamma_{1,i}, \gamma_{2,i}, z_1 - \gamma_{1,i}, z_2 - \gamma_{2,i}} \mid h = \sum_i h_i, \gamma_{j,i} \in (0, z_j), j = 1, 2 \right\}$$

and the corresponding subspaces $\mathcal{CC}_2^{z_1, z_2}(V)$, $\mathcal{CC}_3^{\gamma_1, \gamma_2, \rho_1, \rho_2}(V)$ and $\mathcal{CC}_3^{z_1, z_2}(V)$. Moreover we say that $f \in \mathcal{CC}_1^{\rho_1, \rho_2}$ if

$$\mathcal{N}_{\rho_1, \rho_2}(f) = \|\delta f\|_{\rho_1, \rho_2} + \|\delta_1 f\|_{\rho_1, 0} + \|\delta_2 f\|_{0, \rho_2} + \|f\|_\infty < +\infty \quad (30)$$

with

$$\|\delta_1 f\|_{\rho_1, 0} = \sup_{(s_1, s_2, t_1) \in [0, T]^3} \frac{|\delta_1 f_{s_1 s_2 t_1}|}{|s_2 - s_1|^{\rho_1}}$$

and similar definition in the second direction. The main feature of the space $\mathcal{CC}_2^{z_1, z_2}$ is that $\mathcal{CC}_2^{z_1, z_2} \cap \ker \delta_1 = \{0\}$ if $z_1 > 1$ and $\mathcal{CC}_2^{z_1, z_2} \cap \ker \delta_2 = \{0\}$ if $z_2 > 1$. This implies that the equation $\delta a = 0$ has only a trivial solution $a = 0$ if we require $a \in \mathcal{CC}_1^{z_1, z_2}$ with $z_1, z_2 > 1$.

Let $\mathcal{CC}_i^{1+}(V) = \cup_{z_1 > 1, z_2 > 1} \mathcal{CC}_i^{z_1, z_2}(V)$, $i = 1, 2$.

Note that we have the isomorphism $\mathcal{CC}_{a,b}^{z_1, z_2}(V) \simeq \mathcal{C}_a^{z_1}(\mathcal{C}_b^{z_2}(V)) \simeq \mathcal{C}_b^{z_2}(\mathcal{C}_a^{z_1}(V))$ for $a, b = 0, 1, 2$ and $z_1, z_2 \geq 0$.

Before stating the main result of this section we introduce two versions of the one-dimensional Λ map of Prop. 2.1, acting on the two different coordinates.

Lemma 3.1. $\Lambda_1 : \mathcal{B}_2 \mathcal{CC}_{3,a}^{w_1, z_2}(V) \rightarrow \mathcal{CC}_{2,a}^{w_1, z_2}(V)$ for $a = 1, 2, 3$ with $w_1 > 1$ such that $\delta_1 \Lambda_1 = 1$ and

$$\|\Lambda_1 h\|_{z_1, w_2} \leq C_{z_1} \|h\|_{z_1, w_2} \quad (31)$$

and an analogous bound for Λ_2 .

Proof. If we fix $s_2, u_2, t_2 \in [0, T]$ we can consider $h^{s_2 u_2 t_2} \in \mathcal{C}_2(V)$ such that

$$(s_1, u_1, t_1) \mapsto h_{s_1, u_1, t_1}^{s_2, u_2, t_2} = h_{(s_1, u_1, t_1)}^{(s_2, u_2, t_2)}$$

and note that $h^{s_2 u_2 t_2} \in \mathcal{B}_2 \mathcal{C}_2^{z_1}(V)$ since $\delta h^{s_1 u_1 t_1} = 0$ so that it is in the range of the one-dimensional Λ of Prop. 2.1 and

$$|(\Lambda h^{s_2, u_2, t_2})_{s_1 t_1}| \leq C_{z_1} \|h^{s_2, u_2, t_2}\|_{z_1} |s_1 - t_1|^{z_1}. \quad (32)$$

Then define Λ_1 as

$$(\Lambda_1 h)_{(t_1, s_1), (t_2, u_2, s_2)} := (\Lambda h^{s_2, u_2, t_2})_{s_1 t_1}$$

and note that the bound (32) implies eq. (31). Proceeding similarly one can prove a similar statement about Λ_2 . \square

Then it holds the analogous of the one-dimensional result:

Proposition 3.2. *There exists a unique map $\Lambda : \mathcal{B}\mathcal{C}\mathcal{C}_2^{1+}(V) \rightarrow \mathcal{C}\mathcal{C}_1^{1+}(V)$ such that $\delta\Lambda = 1$. Moreover if $z_1, z_2 > 1$, $h \in \mathcal{B}\mathcal{C}\mathcal{C}_2^{z_1, z_2}(V)$ then*

$$\|\Lambda h\|_{z_1, z_2} \leq \frac{1}{2^{z_1} - 2} \frac{1}{2^{z_2} - 2} \|h\|_{z_1, z_2}.$$

Proof. Since $h \in \text{Im}\delta$ we have $\delta_1 h = \delta_2 h = 0$. Then let $\Lambda h = \Lambda_1 \Lambda_2 h$ which is meaningful since the $\delta_1 \Lambda_2 h = \Lambda_2 \delta_1 h = 0$ (by linearity) and the requirement on the regularity is satisfied. Then

$$\delta \Lambda h = \delta_2 \delta_1 \Lambda_1 \Lambda_2 h = \delta_2 \Lambda_2 h = h$$

and

$$\|\Lambda h\|_{z_1, z_2} = \|\Lambda_1 \Lambda_2 h\|_{z_1, z_2} \leq C_{z_1} \|\Lambda_2 h\|_{z_1, z_2} \leq C_{z_1} C_{z_2} \|h\|_{z_1, z_2}.$$

Uniqueness depends on the fact that $z_1, z_2 > 1$. Using the uniqueness it is easy to deduce that $\Lambda = \Lambda_2 \Lambda_1$, i.e. the one-dimensional maps commute (when they can be both applied). \square

We can already state an interesting result about integration of “small” biincrements.

Corollary 3.3 (2d integration). *Let $a \in \mathcal{C}\mathcal{C}_{2,2}(V)$ such that $\delta_1 a \in \mathcal{C}\mathcal{C}_{3,2}^{z_1, *}$, $\delta_2 a \in \mathcal{C}\mathcal{C}_{2,3}^{*, z_2}$, $\delta a \in \mathcal{C}\mathcal{C}_{2,2}^{z_1, z_2}$ with $z_1, z_2 > 1$. There exists $f \in \mathcal{C}\mathcal{C}_{1,1}(V)$ such that*

$$\delta f = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)a$$

and

$$\lim_{|\Pi| \rightarrow 0} \sum_{i,j} a_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = (\delta f)_{(t^1, s^1), (t^2, s^2)}$$

where the limit is taken over partitions $\Pi = \{(t_i^1, t_j^2)_{i,j}\}$ of the square $[t^1, t^2] \times [s^1, s^2]$ into boxes whose maximum size $|\Pi|$ goes to zero.

Proof. The required conditions on a ensure that the 1-biincrement

$$h = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)a = a - \Lambda_1 \delta_1 a - \Lambda_2 \delta_2 a + \Lambda \delta a$$

is well defined. By direct computation we have that

$$\delta_1 h = \delta_1(1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)a = (\delta_1 - \delta_1)(1 - \Lambda_2 \delta_2)a = 0$$

and $\delta_2 h = 0$. So h must be in the image of δ , i.e. there exists f such that $h = \delta f$. This proves the first claim.

To prove the convergence of the sums consider the above decomposition

$$a = \delta f + \Lambda_2 \delta_2 a + \Lambda_1 \delta_1 a - \Lambda \delta a$$

written as $a = \delta f + r_1 + r_2 + r$ where

$$r_1 = (1 - \Lambda_1 \delta_1) \Lambda_2 \delta_2 a, \quad r_2 = (1 - \Lambda_2 \delta_2) \Lambda_1 \delta_1 a \quad r = \Lambda \delta a.$$

Note that $r_1 \in \mathcal{C}\mathcal{C}_{2,2}^{*, z_2}$, $\delta_1 r_1 = 0$ and $r_2 \in \mathcal{C}\mathcal{C}_{2,2}^{z_1, *}$ and $\delta_2 r_2 = 0$.

Then let

$$\begin{aligned}
S_\Pi &= \sum_{i,j} a_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} \\
&= \sum_{i,j} (\delta f)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} + \sum_{i,j} (r_1)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} \\
&\quad + \sum_{i,j} (r_2)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} + \sum_{i,j} (r)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)}
\end{aligned}$$

and note that, using the fact that δf is an exact biincrement

$$\sum_{i,j} (\delta f)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = (\delta f)_{(t^1, s^1)(t^2, s^2)}$$

and

$$\sum_{i,j} (r_1)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = \sum_j (r_1)_{(t^1, s^1)(t_{j+1}^2, t_j^2)}$$

since $\delta_1 r_1 = 0$ (i.e. r_1 is an exact increment in the direction 1). In the same way

$$\sum_{i,j} (r_2)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = \sum_i (r_2)_{(t_{i+1}^1, t_i^1)(t^2, s^2)}$$

Then

$$\left| \sum_j (r_1)_{(t^1, s^1)(t_{j+1}^2, t_j^2)} \right| \leq \|r_1\| \sum_j |t_{j+1}^2 - t_j^2|^{z_2} \rightarrow 0$$

and

$$\left| \sum_i (r_2)_{(t_{i+1}^1, t_i^1)(t^2, s^2)} \right| \leq \|r_2\| \sum_i |t_{i+1}^1 - t_i^1|^{z_1} \rightarrow 0$$

and finally

$$\left| \sum_{i,j} (r)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} \right| \leq \|r\| \sum_{i,j} |t_{i+1}^1 - t_i^1|^{z_1} |t_{j+1}^2 - t_j^2|^{z_2} \rightarrow 0$$

as $|\Pi| \rightarrow 0$ which proves our claim. \square

4 Two-dimensional Young theory

Proposition 4.1. *Let $f \in \mathcal{CC}_{1,1}^{\gamma_1, \gamma_2}$, $g \in \mathcal{CC}_{1,1}^{\rho_1, \rho_2}$ with $\gamma_1 + \rho_1 = z_1 > 1$, $\gamma_2 + \rho_2 = z_2 > 1$. Then $\delta f \delta g \in \mathcal{CC}_2^{z_1, z_2} \cap \text{Im} \delta$ and the integral $\iint f dg$ can be defined as*

$$\begin{aligned}
\iint f dg &= (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta g) \\
&= -f \delta g + \Lambda(\delta f \delta g) + \int f d_1 \delta_2 g + \int f d_2 \delta_1 g
\end{aligned} \tag{33}$$

where $\int f d_1 \delta_2 g, \int f d_2 \delta_1 g$ are standard Young integrals. Moreover we can define also the integral $\iint d_1 f d_2 g$ as

$$\iint d_1 f d_2 g = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(\delta_1 f \delta_2 g). \quad (34)$$

and for $\rho_1, \rho_2 > 1/2$ we can define

$$\iint f d_1 g d_2 g = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_1 g \delta_2 g) \quad (35)$$

Corollary 4.2. *Under the hypothesis of the previous proposition, the two-dimensional sums of the increments $f \delta g$ converge:*

$$\lim_{|\Pi_{z,w}| \rightarrow 0} \sum_{i,j} (f \delta g)_{x_i x_{i+1}; y_j y_{j+1}} = \left(\iint f d g \right)_{z,w}. \quad (36)$$

where the partition $\Pi_{z,w}$ is taken on the square (z, w) , $z_i \in \mathbb{R}^2$.

Proposition 4.3. *Under the assumption of the Proposition (4.1) we have that:*

$$\begin{aligned} \iint f d_1 g d_2 g &= (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_2 g \delta_1 g) \\ &= (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_1 g \bullet \delta_2 g) \end{aligned} \quad (37)$$

where $(\delta_1 g \bullet \delta_2 g)_{s_1 s_2 t_1 t_2} = \delta_1 g_{s_1 s_2 t_1} \delta_2 g_{s_1 t_1 t_2}$

Proof. Let $a = f \delta_2 g \bullet \delta_1 g$. By a simple computation we have that $\delta_1 a \in \mathcal{C}_{3,2}^{\gamma_1 + \rho_1, *}$, $\delta_2 a \in \mathcal{C}_{2,3}^{*, \gamma_2, \rho_2}$ and $\delta a \in \mathcal{C}_{3,3}^{\gamma_1 + \rho_1, \gamma_2 + \rho_2}$ then the Corollary (3.3) gives

$$\begin{aligned} (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_2 g \bullet \delta_1 g)_{s_1 s_2 t_1 t_2} &= \lim_{|\Pi| \rightarrow 0} \sum_{\Pi} f_{s_i t_j} \delta_2 g_{s_i t_j t_{j+1}} \delta_1 g_{s_i s_{i+1} t_j} \\ &= \lim_{|\Pi| \rightarrow 0} \sum_{\Pi} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta_2 g_{s_{i+1} t_j t_{j+1}} - f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta g_{s_i s_{i+1} t_j t_{j+1}} \end{aligned} \quad (38)$$

where $\Pi := \{(s_i, t_j)\}_{i,j}$ is a partition of $[s_1, s_2] \times [t_1, t_2]$ and we denote also by $\Pi_1 = (s_i)_i$ and $\Pi_2 = (t_j)_j$ a respectively partition of $[s_1, s_2]$ and $[t_1, t_2]$. Once again by the Corollary (3.3) we have that

$$\lim_{|\Pi| \rightarrow 0} \sum_{\Pi} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta_2 g_{s_{i+1} t_j t_{j+1}} = \left(\iint f d_1 g d_2 g \right)_{s_1 s_2 t_1 t_2}$$

and then the one dimensional Young theory of integration gives

$$\lim_{|\Pi_2| \rightarrow 0} \sum_{\Pi_2} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta g_{s_i s_{i+1} t_j t_{j+1}} = \int_{t_1}^{t_2} f_{s_i t} \delta_1 g_{s_i s_{i+1} t} d_t \delta_1 g_{s_i s_{i+1} t}$$

where $\left| \int_{t_1}^{t_2} f_{s_i t} \delta_1 g_{s_i s_{i+1} t} d_t \delta_1 g_{s_i s_{i+1} t} \right| \lesssim (s_{i+1} - s_i)^{2\rho_1} (t_2 - t_1)^{\rho_2}$. Finally using the fact that $\rho_1, \rho_2 > 1/2$ gives:

$$\lim_{|\Pi_1| \rightarrow 0} \lim_{|\Pi_2| \rightarrow 0} \sum_{|\Pi|} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta g_{s_i s_{i+1} t_j t_{j+1}} = 0$$

Putting all these last equation together give us the second line of our proposition, the first part is given by the same argument. \square

Proposition 4.4. Let $x \in \mathcal{C}_{1,1}^{\alpha,\beta}$ and $\varphi \in C^4(\mathbb{R})$ then for $\alpha, \beta > 1/2$ the following change of variable formula :

$$\delta\varphi(x) = \iint \varphi'(x)dx + \iint \varphi''(x)d_1x d_2x$$

hold.

Proof. Let $\Pi_1 = (s_i)_i$ a partition of $[s_1, s_2]$ and $\Pi_2 = (t_j)_j$ of $[t_1, t_2]$ then

$$\begin{aligned} \delta\varphi(x)_{s_1 s_2 t_1 t_2} &= \sum_{\Pi_1 \Pi_2} \delta\varphi(x)_{s_i s_{i+1} t_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} \left(\delta_2 \int_1 \varphi'(x) d_1 x \right)_{s_i s_{i+1} t_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} \int_{s_i}^{s_{i+1}} \delta_2 \varphi'(x)_{s t_j t_{j+1}} d_s x_{s t_{j+1}} + \int_{s_i}^{s_{i+1}} \varphi'(x_{s t_j}) d_s \delta_2 x_{s t_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} a_{ij} + b_{ij} \end{aligned}$$

where we have used the one dimensional change of variable formula for the Young integral. Now let us treat the first term of our sum

$$\begin{aligned} a_{ij} &= \int_{s_i}^{s_{i+1}} \delta_2 \varphi'(x)_{s t_j t_{j+1}} d_s x_{s t_{j+1}} \\ &= \delta_2 \varphi'(x)_{s_i t_j t_{j+1}} \delta_1 x_{s_i s_{i+1} t_j t_{j+1}} + \Lambda_1 (\delta \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} \\ &= \left(\int_{t_j}^{t_{j+1}} \varphi''(x_{s_i t}) d_t x_{s_i t} \right) \delta_1 x_{s_i s_{i+1} t_j t_{j+1}} \\ &\quad + \Lambda_2 \delta_2 (1 - \Lambda_1 \delta_1) (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} - \Lambda \delta (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} \end{aligned}$$

Is not difficult to see that

$$\lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} \Lambda_2 \delta_2 (1 - \Lambda_1 \delta_1) (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} = 0 \quad (39)$$

and

$$\lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} \Lambda \delta (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} = 0 \quad (40)$$

then is remind to treat the term $\left(\int_{t_j}^{t_{j+1}} \varphi''(x_{s_i t}) d_t x_{s_i t} \right) \delta_1 x_{s_i s_{i+1} t_j t_{j+1}}$ and for that we have the following expansion :

$$\left(\int_{t_j}^{t_{j+1}} \varphi''(x_{s_i t}) d_t x_{s_i t} \right) \delta_1 x_{s_i s_{i+1} t_j t_{j+1}} = \varphi''(x_{s_i t_j}) \delta_2 x_{s_i t_j t_{j+1}} \delta_1 x_{s_i s_{i+1} t_j} + r_{s_i t_j t_{j+1}}^b \delta_1 x_{s_i s_{i+1} t_j} \quad (41)$$

where $r^b = \delta_2 \varphi'(x) - \varphi''(x) \delta_2 x \in \mathcal{C}_{1,2}^{*,2\beta}$ such that $\delta_1 r^b \in \mathcal{C}_{2,2}^{\alpha,2\beta}$ which gives :

$$\lim_{|\Pi_1| \rightarrow 0} \sum_{\Pi_1} r_{s_i t_j t_{j+1}}^b \delta_1 x_{s_i s_{i+1} t_j} = \int_{s_1}^{s_2} r_{s t_j t_{j+1}}^b d_s x_{s t_{j+1}}$$

where the limit satisfy the upper bound $|\int_{s_1}^{s_2} r_{st_j t_{j+1}}^b d_s x_{st_{j+1}}| \lesssim_{\alpha, \beta} (s_2 - s_1)^\alpha (t_2 - t_1)^{2\beta}$ these allow us to conclude that :

$$\lim_{|\Pi_2| \rightarrow 0} \lim_{|\Pi_1| \rightarrow 0} \sum_{\Pi_1 \Pi_2} r_{s_i t_j t_{j+1}}^b \delta_1 x_{s_i s_{i+1} t_j} = 0. \quad (42)$$

Then putting together equation (39) , (40) , (41) and (42) we obtain that :

$$\left(\iint \varphi''(x) d_1 x d_2 x \right)_{s_1 s_2 t_1 t_2} = \lim_{|\Pi_2| \rightarrow 0} \lim_{|\Pi_1| \rightarrow 0} \sum_{\Pi_1 \Pi_2} a_{ij}$$

the second sum $\sum_{\Pi_1 \Pi_2} b_{ij}$ is more simple to compute indeed :

$$\begin{aligned} \sum_{\Pi_1 \Pi_2} b_{ij} &= \sum_{\Pi_1 \Pi_2} \int_{s_i}^{s_{i+1}} \varphi'(x_{st_j}) d_s \delta_2 x_{st_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} \varphi'(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} + \Lambda_2 \delta_2 (1 - \Lambda_1 \delta_1) (\varphi'(x) \delta x) - \Lambda \delta (\varphi'(x) \delta x) \end{aligned}$$

and then

$$\begin{aligned} \lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} b_{ij} &= \lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} \varphi'(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} \\ &= \left(\iint \varphi'(x) dx \right)_{s_1 s_2 t_1 t_2} \end{aligned}$$

and this gives the needed result. \square

We have also this immediate generalization for the Young integral.

Proposition 4.5. *Let $y \in \mathcal{C}_{1,1}^{\gamma_1, \gamma_2}, x \in \mathcal{C}_{1,1}^{\rho_1, \rho_2}$ and $z \in \mathcal{C}_{1,1}^{\beta_1, \beta_2}$ such that $\gamma_i + \rho_i > 1, \beta_i + \gamma_i > 1$ and $\beta_i + \rho_i > 1$ for $i = 1, 2$ then we can define*

$$\begin{aligned} \iint y d_1 x d_2 z &= (1 - \Lambda_1 \delta_1) (1 - \Lambda_2 \delta_2) (y \delta_1 x \delta_2 z) \\ \iint y d_2 z d_1 x &= (1 - \Lambda_1 \delta_1) (1 - \Lambda_2 \delta_2) (y \delta_2 z \delta_1 x) \end{aligned}$$

and

$$\iint y d_1 x \bullet d_2 z = (1 - \Lambda_1 \delta_1) (1 - \Lambda_2 \delta_2) (y \delta_1 x \bullet \delta_2 z)$$

and we have the following identity

$$\iint y d_1 x d_2 z = \iint y d_2 z d_1 x = \iint y d_1 x \bullet d_2 z$$

Moreover we have the following Riemann sums representation for our integrals

$$\begin{aligned} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{st} d_s x_{st} d_t z_{st} &= \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta_2 z_{s_{i+1} t_j t_{j+1}} \\ &= \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta_2 z_{s_i t_j t_{j+1}} \\ &= \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_2 z_{s_i t_j t_{j+1}} \delta_1 x_{s_{i+1} t_j t_{j+1}} \end{aligned}$$

Proof. Let $a^1 = y\delta_1 x \delta_2 z$, $a^2 = y\delta_2 z \delta_1 x$ and $a^3 = y\delta_1 x \bullet \delta_2 z$ then by a simple computation we have that

$$\delta_1 a^1 = -\delta_1 y \delta_1 x \delta_2 z - y \delta_1 x \delta z \in \mathcal{CC}_{3,2}^{\min(\gamma_1+\rho_1, \rho_1+\beta_1), \beta_2}$$

$$\delta_2 a^1 = -\delta_2 y \delta_1 x \delta_2 z - y \delta x \delta_2 z \in \mathcal{CC}_{2,3}^{\rho_1, \min(\gamma_2+\beta_2, \rho_2+\beta_2)}$$

and

$$\delta a^1 = \delta y \delta_1 x \delta_2 z + \delta_1 y \delta x \delta_2 z + \delta_2 y \delta_1 x \delta z + y \delta x \delta z \in \mathcal{CC}_{3,3}^{\min(\gamma_1+\rho_1, \rho_1+\beta_1), \min(\gamma_2+\beta_2, \rho_2+\beta_2)}$$

then a^1 satisfies the assumption of the Corollary (3.3) which also true for a^2 and a^3 by a similar computation. Then the integral $\iint y d_1 x d_2 z$, $\iint y d_2 z d_1 x$ and $\iint y d_1 x \bullet d_2 z$ are well defined and we have that

$$(\iint y d_2 z d_1 x)_{s_1 s_2 t_1 t_2} = \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta_2 z_{s_{i+1} t_j t_{j+1}}$$

$$(\iint y d_2 z d_1 x)_{s_1 s_2 t_1 t_2} = \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_2 z_{s_i t_j t_{j+1}} \delta_1 x_{s_i s_{i+1} t_{j+1}}$$

and

$$(\iint y d_1 x \bullet d_2 z)_{s_1 s_2 t_1 t_2} = \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta_2 z_{s_i t_j t_{j+1}}$$

To prove that this last integral coincide it suffices to show that the difference between the Riemann sum vanish when the mesh of the partition go to zero. In fact we have that

$$\begin{aligned} \lim_{|\pi_2| \rightarrow 0} \sum_{i,j} (a^1 - a^3)_{s_i s_{i+1} t_j t_{j+1}} &= \lim_{|\pi_2| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta z_{s_i s_{i+1} t_j t_{j+1}} \\ &= \sum_i \int_{t_1}^{t_2} y_{s_i t} \delta_1 x_{s_i s_{i+1} t} dt \delta_1 z_{s_i s_{i+1} t} \end{aligned} \quad (43)$$

Using the fact that $|\int_{t_1}^{t_2} y_{s_i t} \delta_1 x_{s_i s_{i+1} t} dt \delta_1 z_{s_i s_{i+1} t}| \lesssim (s_{i+1} - s_i)^{\rho_1 + \beta_1}$ we obtain that :

$$\lim_{|\pi_1| \rightarrow 0} \lim_{|\pi_2| \rightarrow 0} \sum_{i,j} (a^1 - a^3)_{s_i s_{i+1} t_j t_{j+1}} = 0$$

which gives the following equality $\iint y d_1 x d_2 z = \iint y d_1 x \bullet d_2 z$. The other identity is obtained by exactly the same computations. \square

5 Analysis of a two-parameter integral

Following the informal approach of sec. 2.1.2 we would like to get inspired for a general construction from the *analysis* of a concrete two-dimensional integral. Then consider $\iint \varphi(x) dx$ for a smooth surface $x : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a smooth function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. The decomposition of eq.(27) gives

$$\iint \varphi(x) dx = -\varphi(x) \int dx + \int_1 \varphi(x) \int_2 dx + \int_2 \varphi(x) \int_1 dx + \iint d\varphi(x) dx$$

where, for example, we used the notation $\int_1 \varphi(x) \int_2 dx$ to mean

$$\left(\int_1 \varphi(x) \int_2 dx \right)_{(s_1, s_2; t_1, t_2)} := \int_{s_1}^{t_1} \varphi(x_{u_1, s_2}) \int_{s_2}^{t_2} \partial_1 \partial_2 x(u_1, u_2) du_1 du_2$$

and the others analogous expressions.

This decomposition is at the origin of Prop. 4.1 when the demanded regularity is satisfied since the iterated integral is given by the formula

$$\iint d\varphi(x) dx = -\Lambda [\delta\varphi(x) \delta x].$$

To proceed further we note that

$$\iint d\varphi(x) = \iint \varphi'(x) dx + \iint \varphi''(x) (d_1 x d_2 x) \quad (44)$$

(recall that $d = d_1 d_2$ is not a derivation). Take the first term in the r.h.s. and use again eq. (27) to write

$$\iint \varphi'(x) dx = -\varphi'(x) \int dx + \int_1 \varphi'(x) \int_2 dx + \int_2 \varphi'(x) \int_1 dx + \iint d\varphi'(x) dx \quad (45)$$

and a similar equation for the term $\iint \varphi''(x) d_1 x d_2 x$. Then

$$\iint d\varphi(x) dx = \iint \varphi'(x) dx dx + \iint \varphi''(x) (d_1 x d_2 x) dx. \quad (46)$$

For simplicity we treat explicitly the first term in the r.h.s, the analysis of the second being similar.

Using eq. (45) and the definition of iterated integral, we have

$$\iint \varphi'(x) dx dx = -\varphi'(x) \int dx dx + \int_1 \varphi'(x) \int_2 dx dx + \int_2 \varphi'(x) \int_1 dx dx + \iint d\varphi'(x) dx dx \quad (47)$$

This expression seems complicated, however it shows that, in order to control the l.h.s. we need two ingredients:

- 1) Being able to define *essentially one-dimensional integrals* like

$$\int_1 \varphi(x) \int_2 dx, \int_1 \varphi'(x) \int_2 dx dx, \int_1 \varphi''(x) \int_2 d_1 x d_2 dx, \dots \quad (48)$$

- 2) Control the *remainders* given by the three-fold iterated integrals

$$\mathcal{R} := \iint d\varphi'(x) dx dx \quad \tilde{\mathcal{R}} := \iint d\varphi''(x) d_1 x d_2 x dx.$$

5.0.1 The boundary integrals

We call integrals like those appearing in eq.(48) *boundary integrals* to emphasize their one-dimensional nature (which, we hope, will be clear from what follows). Their appearance is characteristic of the multidimensional setting and it is linked to the cohomological structure of the complex (\mathcal{CC}_*, δ) studied in Sec. 3. Take the first of them and expand it according to the (one-dimensional) eq.(10):

$$\int_1 \varphi'(x) \int_2 dx dx = \varphi'(x) \iint dx dx + \int_1 d_1 \varphi'(x) \int_2 dx dx \quad (49)$$

Next, apply δ_1 to the second term in the r.h.s.:

$$-\delta_1 \int_1 d_1 \varphi'(x) \int_2 dx dx = \int_1 d_1 \varphi'(x) \int_1 \int_2 dx dx + \int_1 d_1 \varphi'(x) \int_2 dx \int_1 dx \quad (50)$$

The first term in the r.h.s. is *simple* since it is equivalent to $\delta_1 \varphi'(x) \iint dx dx$, and can be controlled with some regularity of φ' and the a-priori knowledge of $\iint dx dx$. The second needs to be further expanded as

$$\begin{aligned} \int_1 d_1 \varphi'(x) \int_2 dx \int_1 dx &= \int_1 \varphi''(x) d_1 x \int_2 dx \int_1 dx \\ &= \varphi''(x) \int_1 d_1 x \int_2 dx \int_1 dx + \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \int_1 dx \end{aligned} \quad (51)$$

Again, the first term in the r.h.s do not pose any further problem so we continue to study only the last one. Set

$$\mathcal{A}_1 := \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \int_1 dx$$

We apply to \mathcal{A}_1 the splitting operator S_1 obtaining

$$S_1 \mathcal{A}_1 = \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \otimes_1 \int_1 dx \quad (52)$$

Let us clarify the meaning of this last term: in the first direction is the tensor product of two 1-increments, while in the second direction is a 1-increment. Taking four points (u_1, u_2, u_3, u_4) in direction 1 and two (v_1, v_2) in direction 2, its value is given by the expression

$$\int_{u_3}^{u_4} \int_{v_1}^{v_2} \left(\int_{u_1}^{u_2} \int_{v_1}^v \left(\int_{u_1}^u \int_{u_1}^a d_b \varphi''(x_{bv_1}) d_a x_{av_1} \right) d_{uv} x_{uv} \right) d_{st} x_{st}$$

As the reader can easily check, by setting $u_2 = u_3$ we reobtain \mathcal{A}_1 .

Denote with $\delta_1 \otimes_1 1$ the action of the δ_1 operator on the first factor of an element of $\mathcal{C}_1 \otimes \mathcal{C}_1$, where again the 1 as index of the tensor operation denote that it acts on tensor products according to the first direction. Apply $\delta_1 \otimes_1 1$ to the r.h.s of eq. (52) last term in the r.h.s to obtain

$$\begin{aligned} -(\delta_1 \otimes_1 1) \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \otimes_1 \int_1 dx &= - \left[\delta_1 \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \right] \otimes_1 \int_1 dx \\ &= \int_1 d_1 \varphi''(x) \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx + \int_1 d_1 \varphi''(x) d_1 x \int_1 \int_2 dx \otimes_1 \int_1 dx \end{aligned}$$

Hence let

$$\mathcal{A}\mathcal{A}_1 := \int_1 d_1 \varphi''(x) \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx = \delta_1 \varphi''(x) \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx$$

and (cfr. eq. (17))

$$\mathcal{A}\mathcal{B}_1 := \int_1 d_1 \varphi''(x) d_1 x \int_1 \int_2 dx \otimes_1 \int_1 dx = (\delta_1 \varphi''(x) - \varphi''(x) \delta_1 x) \int_1 \int_2 dx \otimes_1 \int_1 dx$$

which depend only on the function $\varphi''(x)$ and on the “splitted” iterated integrals

$$\int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx, \quad \int_1 \int_2 dx \otimes_1 \int_1 dx$$

Since we are working under smoothness conditions is possible to recover \mathcal{A}_1 by applying $\Lambda_1 \otimes_1 1$: $\mathcal{A}_1 = -(\Lambda_1 \otimes_1 1)(\mathcal{A}\mathcal{A}_1 + \mathcal{A}\mathcal{B}_1)$. Moreover

$$\mathcal{C}_1 := \int_1 d_1 \varphi''(x) d_1 x = \varphi''(x) \int_1 d_1 x \int_2 dx \int_1 dx + \mu_1 \mathcal{A}_1$$

where recall that μ_1 is the multiplication in the first direction which is the inverse operation of the splitting S_1 . Then we can recover also the term $\int_1 d_1 \varphi'(x) \int_2 dx \int_1 dx$ by eq.(49) and an application of Λ_1 . Finally we have obtained the following expression for the boundary term:

$$\int_1 \varphi'(x) \int_2 dx dx = \varphi'(x) \iint dx dx - \Lambda_1 \left[\delta_1 \varphi'(x) \int_1 \int_2 dx dx + \mathcal{C}_1 \right]$$

where the r.h.s. depends only on a finite number of iterated integrals of x .

5.0.2 The remainders $\mathcal{R}, \tilde{\mathcal{R}}$

The three-fold iterated integral $\iint d\varphi'(x) dx dx$ will be analyzed in terms of its image under δ :

$$\begin{aligned} \delta \iint d\varphi'(x) dx dx &= \iint d\varphi'(x) \iint dx dx + \iint d\varphi'(x) dx \iint dx \\ &\quad + \iint d\varphi'(x) \int_1 dx \int_2 dx + \iint d\varphi'(x) \int_2 dx \int_1 dx \end{aligned} \tag{53}$$

The first term is readily controlled in the same spirit as above. Consider the fourth (the third is similar) which we will write as

$$\begin{aligned} \iint d\varphi'(x) \int_2 dx \int_1 dx &= \int_1 d_1 [\delta_2 \varphi'(x)] \int_2 dx \int_1 dx \\ &= \int_1 \delta_2 [\varphi''(x) d_1 x] \int_2 dx \int_1 dx \\ &= \int_1 \delta_2 [\varphi''(x)] d_1 x \int_2 dx \int_1 dx + \int_1 \varphi''(x) d_1 \delta_2 x \int_2 dx \int_1 dx \\ &= \mathcal{D}_1 + \mathcal{E}_1 \end{aligned} \tag{54}$$

Expanding the integral of $\delta_2[\varphi''(x)]d_1x$ we get

$$\begin{aligned}\mathcal{D}_1 &= \int_1 \delta_2[\varphi''(x)]d_1x \int_2 dx \int_1 dx \\ &= \delta_2[\varphi''(x)] \int_1 d_1x \int_2 dx \int_1 dx + \int_1 d_1 \delta_2[\varphi''(x)]d_1x \int_2 dx \int_1 dx\end{aligned}\quad (55)$$

Again we apply $(\delta_1 \otimes_1 1)S_1$ to the second term:

$$\begin{aligned}& -(\delta_1 \otimes_1 1)S_1 \int_1 d_1 \delta_2[\varphi''(x)]d_1x \int_2 dx \int_1 dx \\ &= \int_1 d_1 \delta_2[\varphi''(x)] \int_1 d_1x \int_2 dx \otimes_1 \int_1 dx + \int_1 d_1 \delta_2[\varphi''(x)]d_1x \int_1 \int_2 dx \otimes_1 \int_1 dx \\ &=: \mathcal{DA}_1 + \mathcal{DB}_1\end{aligned}\quad (56)$$

From which we get:

$$\mathcal{D}_1 = \delta_2[\varphi''(x)] \int_1 d_1x \int_2 dx \int_1 dx - \mu_1[\Lambda_1 \otimes_1 1](\mathcal{DA}_1 + \mathcal{DB}_1)\quad (57)$$

And in the same way:

$$\mathcal{E}_1 = \varphi''(x) \int_1 d_1 \delta_2x \int_2 dx \int_1 dx - \mu_1[\Lambda_1 \otimes_1 1](\mathcal{EA}_1 + \mathcal{EB}_1)\quad (58)$$

with

$$\mathcal{EA}_1 := \delta_1[\varphi''(x)] \int_1 d_1 \delta_2x \int_2 dx \otimes_1 \int_1 dx$$

and

$$\mathcal{EB}_1 := \int_1 d_1 \varphi''(x) d_1 \delta_2x \int_1 \int_2 dx \otimes_1 \int_1 dx.$$

Note that

$$\begin{aligned}\mathcal{F}_1 &:= \mathcal{DB}_1 + \mathcal{EB}_1 \\ &= \delta_2 \left[\int_1 d_1 \varphi''(x) d_1x \right] \int_1 \int_2 dx \otimes_1 \int_1 dx \\ &= \delta_2 [\delta_1 \varphi'(x) - \varphi''(x) \delta_1x] \int_1 \int_2 dx \otimes_1 \int_1 dx.\end{aligned}\quad (59)$$

Together eq.(54), (59) (60) (59) imply the representation:

$$\begin{aligned}\iint d\varphi'(x) \int_2 dx \int_1 dx &= \delta_2[\varphi''(x)] \int_1 d_1x \int_2 dx \int_1 dx + \varphi''(x) \int_1 d_1 \delta_2x \int_2 dx \int_1 dx \\ &\quad - \mu_1[\Lambda_1 \otimes_1 1](\mathcal{DA}_1 + \mathcal{EA}_1 + \mathcal{F}_1).\end{aligned}\quad (60)$$

The only term, appearing in \mathcal{R} which is left is the second term in eq.(53): $\iint d\varphi'(x)dx \iint dx$. This term can be handled together a similar term appearing in the expansion of \mathcal{R} which reads:

$$\iint d\varphi''(x) d_1x d_2x \iint dx$$

(all the other terms in $\tilde{\mathcal{R}}$ are handled as above). Indeed we have that the sum of $\iint d\varphi'(x)dx$ and $\iint d\varphi''(x)d_1xd_2x$ which are nontrivial two-dimensional iterated integrals, appear in the expansion for $\delta\varphi(x)$:

$$\begin{aligned}\delta\varphi(x) = & -\varphi'(x) \iint dx + \int_1 \varphi'(x) \int_2 dx + \int_2 \varphi'(x) \int_1 dx \\ & - \varphi''(x) \iint d_1xd_2x + \int_1 \varphi''(x) \int_2 d_1xd_2x + \int_2 \varphi''(x) \int_1 d_1xd_2x \\ & + \iint d\varphi'(x)dx + \iint d\varphi''(x)d_1xd_2x\end{aligned}\quad (61)$$

In this expression all the terms, except the last two can be expressed, following the approach we used for the boundary integrals above, as functional of a small number of integrals of x .

5.1 Rough sheet

We have shown in this section how the 2-dimensional integral $\iint \varphi(x)dx$ admits to be expressed as a well behaved functional F of a family \mathbb{X} of iterated integrals of x and of $\varphi(x)$. This functional can then be extended to more irregular functions x , not necessarily smooth, in two ways:

- a) *Algebraic approach.* We are given a family \mathbb{X} of biincrements which satisfy algebraic conditions analogous to that which allowed us to perform the computations in this section. In this case, the integral can be defined by the same functional F . The algebraic relations are then needed to show that such a definition is consistent with our notion of integral (e.g. that this integral is in the kernel of both δ_1 and δ_2).
- b) *Geometric approach.* We are able to show that, there exists a sequence of families \mathbb{X}^n obtained by iterated integrals over smooth 2-dimensional functions x^n , which converges, under suitable topologies on the biincrements, to a limiting family \mathbb{X} . Then by the continuity of the map F we are able to identify the limit of $\iint \varphi(x^n)dx^n$ and to consider it as an extension of the integral over smooth 2-parameter functions. This is analogous to the *geometric* theory of rough paths.

Already in the one-dimensional theory the two approaches can give different notions of integrals.

5.1.1 Algebraic assumption and Boundary integrals

Hypothesis 5.1. *If $a = 1, 2$ then $\hat{a} = 2, 1$. Assume that $\alpha, \beta > 1/3$. Now as in the one parameter case we assume the existence of some algebraic object*

$$\begin{aligned}A^x, A^\omega &\in \mathcal{C}_{2,2}^{\alpha,\beta}; \quad B_1^{xx}, B_1^{x\omega} \in \mathcal{C}_{2,2}^{2\alpha,\beta}; \quad B_2^{xx}, B_2^{x\omega} \in \mathcal{C}_{2,2}^{\alpha,2\beta} \\ C^{xx}, C^{\omega x}, C^{x\omega}, C^{\omega\omega} &\in \mathcal{C}_{1,1}^{2\alpha,2\beta}; \quad D_1^{xx}, D_1^{\omega x}, D_1^{\omega\omega} \in (\mathcal{C}_2^\alpha \otimes_1 \mathcal{C}_2^\alpha)(\mathcal{C}_1^{2\beta}), \quad D_2^{xx}, D_2^{\omega x}, D_2^{\omega\omega} \in (\mathcal{C}_2^\beta \otimes_2 \mathcal{C}_2^\beta)(\mathcal{C}_1^{2\alpha}) \\ E_1^{xxx}, E_1^{x\omega x}, E_1^{xx\omega}, E_1^{x\omega\omega} &\in (\mathcal{C}_2^{2\alpha} \otimes_1 \mathcal{C}_1^\alpha)(\mathcal{C}_1^{2\beta}); \quad F_1^{xxx}, F_1^{x\omega x}, F_1^{x\omega\omega}, F_1^{xx\omega} \in (\mathcal{C}_2^{2\alpha} \otimes_1 \mathcal{C}_2^\alpha)(\mathcal{C}_1^{3\beta}) \\ E_2^{xxx}, E_2^{x\omega x}, E_2^{xx\omega}, E_2^{x\omega\omega} &\in (\mathcal{C}_2^{2\beta} \otimes_2 \mathcal{C}_1^\beta)(\mathcal{C}_1^{2\alpha}); \quad F_2^{xxx}, F_2^{x\omega x}, F_2^{x\omega\omega}, F_2^{xx\omega} \in (\mathcal{C}_2^{2\beta} \otimes_2 \mathcal{C}_2^\beta)(\mathcal{C}_1^{3\alpha})\end{aligned}$$

satisfying the following equations

1. $A^x = \delta x$,

2. $\delta_a B_a^{xx} = (\delta_a x) A^x$, $\delta_{\hat{a}} B_a^{xx} = -\mu_{\hat{a}} D_{\hat{a}}^{xx}$
3. $\delta_a C^{xx} = \mu_a D_a^{xx}$
4. $(1 \otimes_a \delta_a) D_a^{xx} = 0$, $\delta_{\hat{a}} D_a^{xx} = A^x \otimes_a A^x$,
5. $(1 \otimes_a \delta_a) E_a^{xxx} = 0$, $(\delta_a \otimes_a 1) E^{xxx} = \delta_a x \otimes D_a^{xx}$, $\delta_{\hat{a}} E_a^{xxx} = F_a^{xxx} + B_a^{xx} \otimes_a A^x$,
6. $\delta_a A^\omega = 0$
7. $\delta_a B_a^{x\omega} = (\delta_a x) A^\omega$, $\delta_{\hat{a}} B_a^{x\omega} = \mu_{\hat{a}} D_a^{x\omega}$
8. $\delta_a C^{\omega x} = \mu_a D_a^{\omega x}$
9. $(1 \otimes_a \delta_a) D_a^{\omega x} = 0$, $\delta_{\hat{a}} D_a^{\omega x} = A^\omega \otimes_a A^x$
10. $(1 \otimes_a \delta_a) E_a^{x\omega x} = 0$, $(\delta_a \otimes 1) E^{x\omega x} = \delta_a x \otimes_a D_a^{\omega x}$, $\delta_{\hat{a}} E_a^{x\omega x} = F_a^{x\omega x} + B_a^{x\omega} \otimes_a A^\omega$
11. $\delta_a C^{x\omega} = \mu_a D_a^{x\omega}$
12. $(1 \otimes_a \delta_a) D_a^{x\omega} = 0$, $\delta_{\hat{a}} D_a^{x\omega} = A^x \otimes_a A^\omega$,
13. $(1 \otimes_a \delta_a) E_a^{xx\omega} = 0$, $(\delta_a \otimes_a 1) E^{xx\omega} = \delta_a x \otimes D_a^{x\omega}$, $\delta_{\hat{a}} E_a^{xx\omega} = F_a^{xx\omega} + B_a^{xx} \otimes_a A^\omega$,
14. $\delta_a C^{\omega\omega} = \mu_a D_a^{\omega\omega}$
15. $(1 \otimes_a \delta_a) D_a^{\omega\omega} = 0$, $\delta_{\hat{a}} D_a^{\omega\omega} = A^\omega \otimes_a A^\omega$,
16. $(1 \otimes_a \delta_a) E_a^{x\omega\omega} = 0$, $(\delta_a \otimes_a 1) E^{x\omega\omega} = \delta_a x \otimes D_a^{\omega\omega}$, $\delta_{\hat{a}} E_a^{x\omega\omega} = F_a^{x\omega\omega} + B_a^{x\omega} \otimes_a A^\omega$,

Remark 5.2. When x is a smooth sheet we can choose this algebraic object as the following iterated integral :

1. $(A^x)_{s_1 s_2 t_1 t_2} = (\iint dx)_{s_1 s_2 t_1 t_2} = \delta x_{s_1 s_2 t_1 t_2}$
2. $(A^\omega)_{s_1 s_2 t_1 t_2} = (\iint d\omega)_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} d_s x_{st} d_t x_{st}$
3. $(B_1^{xx})_{s_1 s_2 t_1 t_2} = (\int_1 d_1 x \int_2 dx)_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 s_{t_1}} d_{st} x_{st}$
4. $(B_1^{x\omega})_{s_1 s_2 t_1 t_2} = (\int_1 d_1 x \int_2 d\omega)_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 s_{t_1}} d_s x_{st} d_t x_{st}$
5. $(C^{xx})_{s_1 s_2 t_1 t_2} = (\iint dx dx)_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(s, t)} d_{rr'} x_{rr'} \right) d_{st} x_{st}$
6. $(C^{\omega x})_{s_1 s_2 t_1 t_2} = (\iint d\omega dx)_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(s, t)} d_r x_{rr'} d_{r'} x_{rr'} \right) d_{st} x_{st}$
7. $(D_1^{xx})_{s_1 s_2 s_3 s_4 t_1 t_2} = (\int_2 \int_1 dx \otimes_1 \int_1 dx)_{s_1 s_2 s_3 s_4 t_1 t_2} = \iint_{(s_3, t_1)}^{(s_4, t_2)} \left(\iint_{(s_1, t_1)}^{(s_2, t)} d_{rr'} x_{rr'} \right) d_{st} x_{st}$
8. $(E_1^{xxx})_{s_1 s_2 s_3 s_4 t_1 t_2} = (\int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx)_{s_1 s_2 s_3 s_4 t_1 t_2} = \iint_{(s_3, t_1)}^{(s_4, t_2)} \left(\iint_{(s_1, t_1)}^{(s_2, t)} \delta_1 x_{s_1 r t_1} d_{rr'} x_{rr'} \right) d_{st} x_{st}$

9.

$$\begin{aligned} (F_1^{xxx})_{s_1 s_2 s_3 s_4 t_1 t_2 t_3} &= \left(\iint dx \int_2 dx \otimes_1 \int_1 dx \right)_{s_1 s_2 s_3 s_4 t_1 t_2 t_3} \\ &= - \iint_{(s_3, t_2)}^{(s_4, t_3)} \left(\iint_{(s_1, t_2)}^{(s_2, t)} \iint_{(s_1, t_1)}^{(r, t_2)} d_{ab} x_{ab} \right) d_{rr'} x_{rr'} d_{st} x_{st} \end{aligned}$$

In this section we assume the previous hypothesis to be true and we will give a "reasonable" construction of the following boundary integrals :

$$\begin{aligned} \int_a y \int_{\hat{a}} dx, \quad \int_a y \int_{\hat{a}} dx dx, \\ \int_a y \int_{\hat{a}} d\omega, \quad \int_a y \int_{\hat{a}} d\omega dx, \end{aligned}$$

which allow us to construct the space of two parameters controlled sheet and integrate them. We begin by recalling the notion of a one dimensional controlled path which the space of the sheet y satisfying the following assumption :

$$\delta_a y = y^{x_a} \delta_a x + y^{\sharp a}, \quad y^{x_a} \in \mathcal{C}_{1,1}^{\alpha,\beta}, \quad (y^{\sharp 1}, y^{\sharp 2}) \in \mathcal{C}_{2,1}^{2\alpha,\beta} \times \mathcal{C}_{1,2}^{\alpha,2\beta}$$

where $x \in \mathcal{C}_{1,1}^{\alpha,\beta}$, $a \in \{1, 2\}$ and we denoted by $\mathcal{Q}_x^{\alpha,\beta}$ this space. Now we will set out a permutation lemma that is useful to conduct the computation in the following.

Lemma 5.3. *We have for $h \in (\mathcal{C}_2 \otimes_a \mathcal{C}_2)(\mathcal{C}_2)$ the following identity:*

$$\delta_a \mu_a h = \mu_a (\delta_a \otimes_a 1) h - \mu_a (1 \otimes_a \delta_a) h. \quad (62)$$

And then we have the following construction for the Boundary integral :

Proposition 5.4. *Assume that the hypothesis 5.1 to be true, and let $y \in \mathcal{Q}_x^{\alpha,\beta}$. Then we define the boundary integral by :*

1. $\int_a y \int_{\hat{a}} dx := y A^x + y^{x_a} B_a^{xx} + \Lambda_a [y^{\sharp a} A^x + \delta_a y^{x_a} B_a^{xx}],$
2. $\int_a y \int_{\hat{a}} dx dx := y C^{xx} + \Lambda_a [\delta_a y C^{xx} + y^x \mu_a E_a^{xxx} + \mu_a (\Lambda_a \otimes_a 1) (y^{\sharp a} D_a^{xx} + \delta_a y^{x_a} E_a^{xxx})],$
3. $\int_a y \int_{\hat{a}} d\omega := y A^\omega + y^{x_a} B_a^{x\omega} + \Lambda_a [y^{\sharp a} A^\omega + y^x B_a^{x\omega}]$
4. $\int_a y \int_{\hat{a}} d\omega dx := y C^{\omega x} + \Lambda_a [\delta_a y C^{\omega x} + y^x \mu_a E_a^{x\omega x} + \mu_a (\Lambda_a \otimes_a 1) (y^{\sharp a} D_a^{\omega x} + \delta_a y^{x_a} E_a^{x\omega x})]$
5. $\int_a y \int_{\hat{a}} dx d\omega := y C^{x\omega} + \Lambda_a [\delta_a y C^{x\omega} + y^x \mu_a E_a^{xx\omega} + \mu_a (\Lambda_a \otimes_a 1) (y^{\sharp a} D_a^{x\omega} + \delta_a y^{x_a} E_a^{xx\omega})]$
6. $\int_a y \int_{\hat{a}} d\omega d\omega := y C^{\omega\omega} + \Lambda_a [\delta_a y C^{\omega\omega} + y^x \mu_a E_a^{x\omega\omega} + \mu_a (\Lambda_a \otimes_a 1) (y^{\sharp a} D_a^{\omega\omega} + \delta_a y^{x_a} E_a^{x\omega\omega})]$

Moreover all these formulas have meaning and when x is differentiable we can choose the rough sheet so that they coincide well with their definition in the Riemann-Stieltjes case, which justifies the notation.

Proof. We will only prove the first two formula, for the others we have identical proofs. Let now x a smooth sheet and $y \in \mathcal{Q}_x^{\alpha, \beta}$, then we have easily the following expansion

$$\begin{aligned} \left(\int_1 y \int_2 dx \right)_{s_1 s_2 t_1 t_2} &:= \iint_{(s_1, t_1)}^{(s_2, t_2)} y_{st_1} d_{st} x_{st} = \int_{s_1}^{s_2} y_{st_1} d_1 \delta_2 x_{st_1 t_2} \\ &= y_{s_1 t_1} (A^x)_{s_1 s_2 t_1 t_2} + y_{s_1 t_1}^x (B_1^{xx})_{s_1 s_2 t_1 t_2} + \int_{s_1}^{s_2} y_{s_1 s t_1}^{\#1} d_s \delta_2 x_{st_1 t_2} \end{aligned}$$

where $A^x = \delta x$ and $(B_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 s t_1} d_{st} x_{st}$. Now is easy to check that $r_{s_1 s_2 t_1 t_2} := \int_{s_1}^{s_2} y_{s_1 s t_1}^{\#1} d_s \delta_2 x_{st_1 t_2} \in \mathcal{C}_{2,2}^{3\alpha, \beta}$ and $\delta_1 r = y^{\#1} \delta x + \delta_1 y^{x_1} B_1^{xx} \in \mathcal{C}_{3,2}^{3\alpha, \beta}$. So finally we obtain :

$$r = \Lambda_1 [y^{\#1} \delta x + \delta_1 y^{x_1} B_1^{xx}]$$

and thus

$$\int_1 y \int_2 dx := y A^x + y^{x_1} B_1^{xx} + \Lambda_1 [y^{\#1} A^x + \delta_1 y^{x_1} B_1^{xx}]$$

Now we remark that this formula is still valid when x satisfy only the assumption (5.1). Now we will focus on the second equation which requires a different argument. A quick computation gives :

$$\begin{aligned} \left(\int_1 y \int_2 dx dx \right)_{s_1 s_2 t_1 t_2} &:= \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} y_{rt_1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &= y_{s_1 t_1} (C^{xx})_{s_1 s_2 t_1 t_2} + \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} \delta_1 y_{s_1 r t_1} d_{rr'} x_{rr'} d_{st} x_{st} \end{aligned}$$

The first term of this last equation is well understood, we will focus on the second term denoted by \mathcal{H} in the sequel. We observed that $\mathcal{H} \in \mathcal{C}_{2,2}^{3\alpha, 2\beta}$, then:

$$\begin{aligned} \delta_1 \mathcal{H}_{s_1 s_2 s_3 t_1 t_2} &= \delta_1 y_{s_1 s_2 t_1} C_{s_2 t_1 s_3 t_2}^{xx} + y_{s_1 t_1}^x \iint_{(s_2, t_1)}^{(s_3, t_2)} \iint_{(s_1, t_1)}^{(s_2, t)} \delta_1 x_{s_1 r t_1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &+ \iint_{(s_2, t_1)}^{(s_3, t)} \iint_{(s_1, t_1)}^{(s_2, t)} y_{s_1 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st}. \end{aligned}$$

Now, if we put

$$\mathcal{A}_{s_1 s_2 s_3 t_1 t_2} := \iint_{(s_2, t_1)}^{(s_3, t)} \iint_{(s_1, t_1)}^{(s_2, t)} y_{s_1 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st}$$

we obtain

$$\begin{aligned} (\delta_1 \otimes_1 1) \mathcal{S}_1(\mathcal{A})_{s_1 s_2 v s_3 s_4 t_1 t_2} &= \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} y_{s_1 r t_1}^{\#1} - y_{s_2 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &= y_{s_1 s_2 t_1}^{\#1} \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} d_{rr'} x_{rr'} d_{st} x_{st} \\ &+ \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} \delta_1 y_{s_1 s_2 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &= y_{s_1 s_2 t_1}^{\#1} \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} d_{rr'} x_{rr'} d_{st} x_{st} \\ &+ \delta_1 y_{s_1 s_2 t_1}^x \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} \delta_1 x_{s_2 r t_1} d_{rr'} x_{rr'} d_{st} x_{st} \end{aligned}$$

Then if we recall that the two last iterated integrals are denoted respectively by D_1^{xx} and E_1^{xxx} we obtain:

$$\mathcal{S}_1 \mathcal{A}_1 = (\Lambda_1 \otimes_1 1)[y^{\sharp 1} D_1^{xx} + \delta_1 y^{x_1} E_1^{xxx}]$$

then

$$\mathcal{H} = \Lambda_1[\delta_1 y C^{xx} + y^{x_1} \mu_1 E_1^{xxx} + \mu_1(\Lambda_1 \otimes_1 1)(y^{\sharp 1} D_1^{xx} + \delta_1 y_1^x E_1^{xxx})]$$

This last equation gives us the second formula when x is a smooth sheet, in the general case when x satisfies only the Hypothesis 5.1. It's easy to see that all terms or we apply Λ_1 enjoy the regularity and thanks to the Lemma 5.3 it also satisfy the required algebraic conditions (i.e. : $\delta_1[y C^{xx} + y^{x_1} \mu_1 E_1^{xxx} + \mu_1(\Lambda_1 \otimes_1 1)(y^{\sharp 1} D_1^{xx} + \delta_1 y_1^x E_1^{xxx})] = 0$) and this concludes the proof. \square

5.1.2 Controlled Sheet

Definition 5.5. Let $x \in \mathcal{C}_{1,1}$ such that $\delta x \in \mathcal{C}_{2,2}^{\alpha,\beta}$, and assume the algebraic hypothesis 5.1 to be true then we define the space of the two parameter controlled sheet $\mathcal{K}_x^{\alpha,\beta}$ by $y \in \mathcal{K}_x^{\alpha,\beta}$ if :

1. $\delta y = -y^x A^x - y^\omega A^\omega + \sum_{a=1,2} (\int_a y^x \int_{\hat{a}} dx + \int_a y^\omega \int_{\hat{a}} d\omega) + y^\sharp$
2. $y, y^x, y^\omega \in \mathcal{Q}^{\alpha,\beta}$

Theorem 5.6. For $\alpha, \beta > 1/3, x \in \mathcal{C}_{1,1}$ such that $\delta x \in \mathcal{C}_{2,2}^{\alpha,\beta}$ and assume that the hypothesis (5.1) is true then for $y \in \mathcal{K}_x^{\alpha,\beta}$ we define the increment $\iint y dx \in \mathcal{C}_{2,2}^{\alpha,\beta}$ and $\iint y d\omega \in \mathcal{C}_{2,2}^{\alpha,\beta}$ by:

$$\iint y dx := -y A^x + y^x C^{xx} + y^\omega C^{\omega x} + \sum_{a=1,2} \left(\int_a y \int_{\hat{a}} dx + \int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) + r^b$$

where

$$r^b = \Lambda \left[\delta y A^x - \delta \left(-y^x C^{xx} - y^\omega C^{\omega x} + \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) \right) \right]$$

and

$$\iint y d\omega := -y A^\omega + y^x C^{x\omega} + y^\omega C^{\omega\omega} + \sum_{a=1,2} \left(\int_a y \int_{\hat{a}} d\omega + \int_a y^x \int_{\hat{a}} dx d\omega + \int_a y^\omega \int_{\hat{a}} d\omega d\omega \right) + r^{b,\omega}$$

with

$$r^{b,\omega} = \Lambda \left[\delta y A^\omega - \delta \left(-y^x C^{x\omega} - y^\omega C^{\omega\omega} + \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx d\omega + \int_a y^\omega \int_{\hat{a}} d\omega d\omega \right) \right) \right]$$

Theses two formula are well defined moreover if x is a smooth sheet then this two definition coincide with that given by the Riemann-Stieltjes theory of integration.

Proof. Let x a differentiable sheet then

$$\iint y dx = -y \delta x + \sum_{a=1,2} \int_a y \int_{\hat{a}} dx + \iint dy dx.$$

Now using the fact that $y \in \mathcal{K}_x^{\alpha, \beta}$ we have

$$\iint dy dx = -y^x C^{xx} - y^\omega C^{\omega x} \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) + \iint y^\# dx.$$

Finally if we remark that $\iint y^\# dx \in \mathcal{CC}_{2,2}^{3\alpha, 3\beta}$ we obtain

$$\iint y^\# dx := \Lambda \left[\delta \iint dy dx - \delta \left(-y^x C^{xx} - y^\omega C^{\omega x} \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) \right) \right]$$

This give us the formula when x is smooth. Now we have to check that last formula have meaning in general case in other word we must show that we can apply Λ for $r := \delta \iint dy dx - \delta(-y^x C^{xx} - y^\omega C^{\omega x} \sum_{a=1,2} (\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx))$, for this we will make some preliminary computation.

$$\delta \iint dy dx = \delta y \delta x$$

$$\begin{aligned} \delta(y^x C^{xx}) &= \delta_2(-\delta_1 y^x C^{xx} + y^x \delta_1 C^{xx}) = \delta y^x C^{xx} + y^x \delta x \delta x - \delta_1 y^x \delta_2 C^{xx} - \delta_2 y^x \delta_1 C^{xx} \\ \delta \int_a d_a y^x \int_{\hat{a}} dx dx &= \delta_{\hat{a}} \left(\delta_a y^x C^{xx} + y^{xx_a} \mu_a E_a^{xxx} + (\Lambda_a \otimes_a 1) \left[\delta_a y^{xx_a} E_a^{xxx} + y^{x\#a} D_a^{xx} \right] \right) \\ &= -\delta y^x C^{xx} + \delta_a y^x \delta_{\hat{a}} C^{xx} - \delta_{\hat{a}} y^{xx_a} \mu_a E_a^{xxx} + y^{xx_a} (B_a^{xx} \delta x + \mu_a F_a^{xxx}) \\ &\quad + \mu_a (\Lambda_a \otimes_a 1) [y^{x\#a} \delta x \otimes_a \delta x + \delta_a y^{xx_a} B_a^{xx} \otimes_a \delta x - \delta_{\hat{a}} y^{x\#a} D_a^{xx} + \delta_a y^{xx_a} F_a^{xxx} \\ &\quad - \delta y^{xx_a} E_a^{xxx}] \\ &= -\delta y^x C^{xx} + \delta_a y^x \delta_{\hat{a}} C^{xx} - \delta_{\hat{a}} y^{xx_a} \mu_a E_a^{xxx} + y^{xx_a} \mu_a F_a^{xxx} + \left(\int_a d_a y^x \int_{\hat{a}} dx \right) \delta x \\ &\quad + (\Lambda_a \otimes_a 1) \left[-\delta_{\hat{a}} y^{x\#a} D_a^{xx} + \delta_a y^{xx_a} F_a^{xxx} - \delta y^{xx_a} E_a^{xxx} \right] \end{aligned}$$

It is easy to see that we have a similar equation if we replace dx by $d\omega := d_1 x d_2 x$. Finally we obtain

$$\begin{aligned} r &= y^\# \delta x + \delta y^x C^{xx} + \delta y^\omega C^{\omega x} + \sum_{a=1,2} (\delta_{\hat{a}} y^{xx_a} \mu_a E_a^{xxx} - y^{xx_a} \mu_a F_a^{xxx} + \delta_{\hat{a}} y^{\omega x_a} \mu_a E_a^{x\omega x} - y^{\omega x_a} \mu_a F_a^{x\omega x} \\ &\quad + (\Lambda_a \otimes_a 1) [\delta_{\hat{a}} y^{x\#a} D_a^{xx} - \delta_a y^{xx_a} F_a^{xxx} + \delta y^{xx_a} E_a^{xxx} + \delta_{\hat{a}} y^{\omega\#a} D_a^{\omega x} - \delta_a y^{\omega x_a} F_a^{x\omega x} + \delta y^{\omega x_a} E_a^{x\omega x}]) \end{aligned}$$

and this allow us to say that $r \in \mathcal{CC}_{3,3}^{3\alpha, 3\beta}$ which finishes the proof. \square

Remark 5.7. We observe that this definition of the two parameter integral is not consistent with the definition of the controlled sheet, indeed if $y \in \mathcal{K}_x^{\alpha, \beta}$ then the element $z \in \mathcal{CC}_{1,1}$ defined by $z_{0t} = z_{s0} = 0$ and $\delta z = \iint y dx$ is not in general a controlled sheet.

5.2 Stability under mapping by regular functions

In this section we show that $\varphi(x) \in \mathcal{K}_x^{\alpha, \beta}$ under more algebraic and geometric assumptions. To prove this result we will proceed by linear approximation the problem is that the terms which contain $d_1 x d_2 x$ does not approximate well to bypass this difficulty we will start by giving an alternative expression for the space $\mathcal{K}_x^{\alpha, \beta}$

Hypothesis 5.8. Let $\alpha, \beta > 1/3$ and $a = 1, 2$. Assume that there exist

$$G_a^{xx} \in \mathcal{C}_{2,2}^{\alpha,\beta}, \quad H_a^{xx} \in \mathcal{C}_{2,2}^{\hat{a}\alpha,a\beta}, \quad I_a^{xx} \in \mathcal{C}_{2,2}^{\hat{a}\alpha,a\beta}, \quad J_a^{xx} \in \mathcal{C}_{2,2}^{2\alpha,2\beta},$$

with the convention if $a = 1$ then $\hat{a} = 2$ and conversely. And we assume that this object satisfy the following relation:

1. $\delta_a G_a^{xx} = \delta_a I_{\hat{a}}^{xx} = 0$
2. $\delta_a H_a^{xx} = G_a^{xx} \delta_a x$
3. $\delta_a J_a^{xx} = I_{\hat{a}}^{xx} \delta_a x$
4. $I_a^{xx} = B_a^{xx} + C^{xx}$
5. $A^\omega = 1/2 \delta x^2 - \iint x dx = G_a^{xx} - I_a^{xx}$
6. $B_a^{x\omega} = H_a^{xx} - J_a^{xx}$

Remark 5.9. In regular case this last iterated integral are given by :

1. $(G_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \delta_2 x_{st_1 t_2} d_s x_{st_2}$
2. $(H_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \delta_1 x_{s_1 s t_1} \delta_2 x_{st_1 t_2} d_s x_{st_2}$
3. $(I_2^{xx})_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \delta_2 x_{st_1 t} d_{st} x_{st}$
4. $(J_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \delta_1 x_{s_1 s t_1} \int_{t_1}^{t_2} \delta_2 x_{st_1 t} d_{st} x_{st}$

Now under these assumption we give an alternative expression for the space $\mathcal{K}_x^{\alpha,\beta}$.

Proposition 5.10. Assume the Hypothesis 5.8 and 5.1 are true then for $y \in \mathcal{Q}_x^{\alpha,\beta}$ we have :

1. $\int_a y \delta_{\hat{a}} x d_a x := y G_a^{xx} + y^{xa} H_a^{xx} + \Lambda_a [y^{\sharp a} G_a^{xx} + \delta_a y^{xa} H_a^{xx}]$
2. $\int_a y \int_{\hat{a}} \delta_{\hat{a}} x dx := y I_{\hat{a}}^{xx} + y^{xa} J_a^{xx} + \Lambda_a [y^{\sharp a} I_{\hat{a}}^{xx} + \delta_a y^{xa} J_a^{xx}]$
3. $\int_a y \int_{\hat{a}} d\omega = \int_a y \delta_{\hat{a}} x d_a x - \int_a y \int_{\hat{a}} \delta_{\hat{a}} x dx$

where 1 and 2 are well defined, moreover we can choose G_a , H_a and J_a such that the rough-integrals 1 and 2 coincide with their definition in the Riemann-Stieltjes case.

Proof. We will only proof the first assertion (the proof of second assertion is similar). We assume that x is smooth then we have :

$$\left(\int_1 y \delta_2 x d_a x \right)_{s_1 s_2 t_1 t_2} := \int_{s_1}^{s_2} y_{st_1} \delta_2 x_{st_1 t_2} d_s x_{st} = y_{s_1 t_1} (G_1^{xx})_{s_1 s_2 t_1 t_2} + y^{x1} (H_1^{xx})_{s_1 s_2 t_1 t_2} + r_{s_1 s_2 t_1 t_2}$$

where

$$r_{s_1 s_2 t_1 t_2} := \int_{s_1}^{s_2} y_{st_1 t_2}^{\sharp 1} \delta_2 x_{st_1 t_2} d_s x_{st_2}$$

Now is clear that $r \in \mathcal{C}_{2,2}^{3\alpha,\beta}$ and :

$$\delta_1 r = y^{\sharp 1} G_1^{xx} + \delta_1 y^{x1} H_1^{xx} \in \mathcal{C}_{3,2}^{3\alpha,\beta}$$

and we get

$$r = \Lambda_1 [y^{\sharp 1} G_1^{xx} + \delta_1 y^{x1} H_1^{xx}].$$

The proof of the last assertion is immediate consequence of the assumption 5 and 6 of the hypothesis 5.8. \square

This proposition allow us to say that $y \in \mathcal{K}_x^{\alpha,\beta}$ if and only if :

1. $y \in \mathcal{Q}_x^{\alpha,\beta}$
2. $\delta y = -y^x \delta x - y^\omega (1/2 \delta x^2 - x \delta x) + \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx + \int_a y^\omega \delta_{\hat{a}} x d_a x \right) + y^\sharp$

where $y^x, y^\omega \in \mathcal{Q}_x^{\alpha,\beta}$, $y^\sharp \in \mathcal{C}_{2,2}^{\alpha,\beta}$. Now let us describe our strategy to prove that $\varphi(x) \in \mathcal{K}_x^{\alpha,\beta}$. Let introduce the approximation:

$$\begin{aligned} x_{st}^1 &= x_{0t} + s(x_{1t} - x_{0t}) \quad , \quad x_{st}^2 = x_{s0} + t(x_{s1} - x_{s0}) \\ x_{st}^{12} &= x_{00} + s(x_{10} - x_{00}) + t(x_{01} - x_{00}) + st \delta x_\square \end{aligned}$$

where the \square is the unit square. Now is clear if $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ then we have

$$\begin{aligned} \delta \varphi(x^{12}) &= -\varphi'(x^{12}) \delta x - \varphi''(x^{12}) (1/2 \delta(x^{12})^2 - x^{12} \delta x^{12}) \\ &\quad + \sum_{a=1,2} \left(\int_a \varphi'(x^{12}) \int_{\hat{a}} dx^{12} + \int_a \varphi''(x^{12}) \delta_{\hat{a}} x^{12} d_a x^{12} \right) + R^{12} \end{aligned} \quad (63)$$

where

$$\begin{aligned} R^{12} &= \iint \varphi''(x^{12}) dx^{12} dx^{12} + \iint \varphi'''(x^{12}) d_1 x^{12} d_2 x^{12} dx^{12} + \iint \varphi'''(x^{12}) dx^{12} d_1 x^{12} d_2 x^{12} \\ &\quad + \iint \varphi^{(iv)}(x^{12}) d_1 x^{12} d_2 x^{12} d_1 x^{12} d_2 x^{12} - \varphi''(x^{12}) \iint dx^{12} dx^{12} - \sum_{a \in \{1,2\}} \int_a d_a \varphi''(x^{12}) \int_{\hat{a}} \delta_{\hat{a}} x^{12} dx^{12} \end{aligned} \quad (64)$$

Now our goal is to give similar formula for x^a and to compare them with the same expansion for x . To do that we need to give a meaning for the boundary integral appearing in the expansion of x^a and for that we construct $B_a^{x^{\hat{a}} x^{\hat{a}}}$, $G_a^{x^{\hat{a}} x^{\hat{a}}}$ and $H_a^{x^{\hat{a}} x^{\hat{a}}}$. But by formal computation we see that

$$(B_2^{x^1 x^1})_{s_1 s_2 t_1 t_2} = (s_2 - s_1) \left(\int_{t_1}^{t_2} \delta_2 x_{0t_1 t} dt \delta_1 x_{01t} + s_1 \int_{t_1}^{t_2} \delta x_{01t_1 t} dt \delta_1 x_{01t} \right)$$

Then we define $B_2^{x^1 x^1}$ by the following formula :

$$(B_2^{x^1 x^1})_{s_1 s_2 t_1 t_2} := (s_2 - s_1) (B_2^{xx} + s_1 (K_2^{xx})_{01t_1 t_2})$$

where

$$(K_2^{xx})_{s_1 s_2 t_1 t_2} := \int_{t_1}^{t_2} \delta x_{s_1 s_2 t_1 t} dt \delta_1 x_{s_1 s_2 t}$$

$$\delta_2(K_2^{xx})_{s_1 s_2 t_1 t_2 t_3} = \delta x_{s_1 s_2 t_1 t_2} \delta x_{s_1 s_2 t_1 t_3}$$

Similar computation allow us to define $(G_2^{x^1, x^1})$ and $H_2^{x^1, x^1}$ in the following way :

$$(G_2^{x^1, x^1})_{s_1 s_2 t_1 t_2} = (s_2 - s_1)((G_2^{xx})_{01 t_1 t_2} + (s_2 - 1)(L_2^{xx})_{01 t_1 t_2})$$

$$(H_2^{x^1, x^1})_{s_1 s_2 t_1 t_2} = (s_2 - s_1)(H_2^{xx})_{01 t_1 t_2} + (s_2 - 1)(M_2^{xx})_{01 t_1 t_2} + s_1((N_2^{xx})_{01 t_1 t_2} + (s_2 - 1)(O_2^{xx})_{01 t_1 t_2}))$$

where

$$(L_2^{xx})_{s_1 s_2 t_1 t_2} := \int_{t_1}^{t_2} \delta_1 x_{s_1 s_2 t} d_t \delta_1 x_{s_1 s_2 t}, \quad (M_2^{xx})_{s_1 s_2 t_1 t_2} := \int_{t_1}^{t_2} \delta x_{s_1 s_2 t_1 t} \delta_1 x_{s_1 s_2 t} d_t x_{s_2 t}$$

and

$$(O_2^{xxx})_{s_1 s_2 t_1 t_2} := \int_{t_1}^{t_2} \delta x_{s_1 s_2 t_1 t} \delta_1 x_{s_1 s_2 t} d_t \delta_1 x_{s_1 s_2 t}$$

and this of course pushes us to give a more algebraic assumption on the sheet x :

Hypothesis 5.11. *Let $\alpha, \beta > 1/3$, $a = 1, 2$ and and assume the existence of :*

$$K_a^{xx}, M_a^{xx} \in \mathcal{CC}_{2,2}^{2\alpha, 2\beta}, \quad L_a^{xx} \in \mathcal{CC}_{2,2}^{a\alpha, \hat{a}\beta}, \quad O_2^{xxx} \in \mathcal{CC}_{2,2}^{3\alpha, 2\beta}, \quad O_1^{xxx} \in \mathcal{CC}_{2,2}^{2\alpha, 3\beta}$$

which satisfies the algebraic relation

1. $\delta_2(K_2^{xx})_{s_1 s_2 t_1 t_2 t_3} = \delta x_{s_1 s_2 t_1 t_2} \delta x_{s_1 s_2 t_1 t_3}$
2. $\delta_2 L_2^{xx} = 0$
3. $\delta_2(M_2^{xx})_{s_1 s_2 t_1 t_2 t_3} = \delta_2 x_{s_1 t_1 t_2} (L_2^{xx})_{s_1 s_2 t_2 t_3}$
4. $\delta_2(N_2^{xx})_{s_1 s_2 t_1 t_2 t_3} = \delta x_{s_1 s_2 t_1 t_2} (G_2^{xx})_{s_1 s_2 t_1 t_2}$
5. $\delta_2(O_2^{xxx})_{s_1 s_2 t_1 t_2} = \delta x_{s_1 s_2 t_1 t_2} (L_2^{xx})_{s_1 s_2 t_2 t_3}$

and same relation for K_1^{xx} , M_1^{xx} , N_1^{xx} and N_1^{xx} .

Remark 5.12. *Under this new hypothesis we have some fact :*

1. $\delta_2 B_2^{x^1, x^1} = \delta x^1 \delta_2 x^2$
2. $\delta_2 G_2^{x^1, x^1} = 0$
3. $\delta_2 H_2^{x^1, x^1} = G_2^{x^1, x^1} \delta_2 x^2$

This new hypothesis allows us to define

$$R^1 = \delta \varphi(x^1) - \left(-\varphi'(x^1) \delta x^1 - \varphi''(x^1) (1/2 \delta(x^1)^2 - x^1 \delta x^1) + \sum_{a=1,2} \left(\int_a \varphi'(x^1) \int_{\hat{a}} dx^1 + \int_a \varphi''(x^1) \delta_{\hat{a}} x^1 d_a x^1 \right) \right) \quad (65)$$

and an analogue formula for R^2 . Then we have the following relation between the remainder terms

Proposition 5.13. Let R^1, R^2 and R^{12} given respectively by the equations (65) and (63) and assume that hypothesis 5.11, 5.8 and 5.1 are true then we have $R_\square = R_\square^1 + R_\square^2 - R_\square^{12}$ where

$$R = \varphi(x) - \left(-\varphi'(x)\delta x - \varphi''(x)(1/2\delta x^2 - x\delta x) + \sum_{a \in \{1,2\}} \int_a \varphi'(x) \int_{\hat{a}} dx + \int_a \varphi''(x)\delta_{\hat{a}}x d_a x \right)$$

Proof. The fact that x^1 and x^2 are respectively smooth in the first and second direction gives

$$\left(\int_1 \varphi'(x^1) \int_2 dx^1 \right)_\square = \left(\int_1 \varphi'(x^{12}) \int_2 dx^{12} \right)_\square, \quad x_{ab}^l = x_{ab}^{12}$$

$$\left(\int_2 \varphi'(x^2) \int_1 dx \right)_\square = \left(\int_2 \varphi'(x^{12}) \int_1 dx^{12} \right)_\square$$

For $(a, b) \in \{0, 1\}^2$ and $l = 1, 2$. And of course similar equation for the boundary integrals given by $\int_1 \varphi''(x^1)\delta_2 x^1 d_2 x^1$. On the other side we have by the definition of the functional Λ_1 that :

$$\begin{aligned} \left(\int_2 \varphi'(x) \int_1 dx \right)_\square &= (1 - \Lambda_1 \delta_1)(\varphi'(x)\delta x + \varphi''(x)B_2^{xx})_\square = (1 - \Lambda^{1d}\delta^{1d})(\varphi'(x_0.)\delta x_{01..} + \varphi''(x_0.)(B_2^{xx})_{01..})_{01} \\ &= (1 - \Lambda^{1d}\delta^{1d})(\varphi'(x_0^1.)\delta x_{01..} + \varphi''(x_0^1.)(B_2^{x^1\tilde{x}^1})_{01..})_{01} \\ &= \left(\int_2 \varphi'(x^1) \int_1 dx^1 \right)_\square \end{aligned}$$

and by similar argument we have also that

$$\left(\int_2 \varphi''(x)\delta_1 x d_2 x \right)_\square = \left(\int_2 \varphi''(x^1)\delta_1 x^1 d_2 \tilde{x}^1 \right)_\square$$

Then putting these equation together we obtain the needed identity. \square

Now to show that $R \in \mathcal{C}_{2,2}^{2\alpha, 2\beta}$ we have to give estimates for the remainder terms R^1 and R^2 . At this point we give three technical lemma which help us to do this.

Lemma 5.14. Let $x \in \mathcal{CC}_{2,2}$ and $\varphi \in C^3(\mathbb{R})$, and we define $\nu_1(x)$ by :

$$\begin{aligned} \nu_1(x) &= -\varphi'(x_{00})\delta x_{0101} - \varphi''(x_{00})(1/2\delta x_{0101}^2 - x_{00}\delta x_{0101}) \\ &\quad + \left(\int_0^1 \varphi'(x_{00} + s\delta_1 x_{010}) ds \right) \delta x_{0101} + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) \delta_2 x_{001} \delta_1 x_{011} \\ &\quad + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) \delta x_{0101} \delta_1 x_{011} \\ &\quad - (\varphi'(x_{10}) - \varphi'(x_{00}) - \varphi''(x_{00})\delta_1 x_{010}) \delta_2 x_{101} \end{aligned}$$

then the following inequality hold

$$|\nu_1(x)| \lesssim \sup_{s \in [0,1]} |\varphi''(x_{s0})|(\delta x_{0101})^2 + \sup_{s \in [0,1]} |\varphi'''(x_{s0})| |\delta_1 x_{010} \delta_2 x_{001} \delta x_{0101}|$$

Proof. We begin by remark that

$$\begin{aligned}
& \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) \delta x_{0101} \delta_1 x_{011} = \\
& \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) (\delta x_{0101})^2 + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) \delta_1 x_{010} \delta x_{0101} \\
& = \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) (\delta x_{0101})^2 + \varphi'(x_{10}) \delta x_{0101} - \left(\int_0^1 \varphi'(x_{00} + s\delta_1 x_{010}) ds \right) \delta x_{0101}
\end{aligned}$$

and

$$\delta x_{0101}^2 = 2x_{00}\delta x_{0101} + 2\delta_1 x_{010}\delta_2 x_{101} + 2\delta_2 x_{001}\delta x_{0101} + (\delta x_{0101})^2$$

injecting these two equality in the definition of $\nu_1(x)$ gives :

$$\begin{aligned}
\nu_1(x) &= -(\varphi'(x_{10}) - \varphi'(x_{00}))\delta_2 x_{001} + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) \delta_1 x_{011} \delta_2 x_{001} - \varphi''(x_{00})\delta_2 x_{001} \delta y_{0101} \\
&+ \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) (\delta x_{0101})^2 - 1/2 \varphi''(x_{00}) (\delta y_{0101})^2 \\
&= \left(\int_0^1 \int_0^1 \varphi'''(x_{00} + ss'\delta_1 x_{010}) s ds' ds \right) \delta_1 x_{010} \delta_2 x_{001} \delta x_{0101} \\
&+ \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) (\delta x_{0101})^2 - 1/2 \varphi''(x_{00}) (\delta x_{0101})^2
\end{aligned}$$

Then the desired inequality is a simple consequence of this equality. \square

Lemma 5.15. *Let $y \in \mathcal{C}_{1,1}$ and $\alpha, \beta > 1/3$ such that $\delta_2 y \in \mathcal{C}_{1,2}^{\alpha,\beta}$, $\delta y \in \mathcal{C}_{2,2}^{\alpha,\beta}$ moreover we assume that y is smooth in the first direction and that there exists $\int_2 \int_2 d_2 y d_2 y \in \mathcal{C}_{1,2}^{\alpha,2\beta}$, $\int_2 \int_2 dy d_2 y \in \mathcal{C}_{2,2}^{\alpha,2\beta}$, $H_2^{yy} = \int_2 \delta_2 y \delta_1 y d_2 y \in \mathcal{C}_{2,2}^{1,2\beta}$, $G_2^{yy} = \int_2 \delta_1 y d_2 y \in \mathcal{C}_{2,2}^{1,\beta}$ and $B_2^{yy} = \int_2 d_2 y \int_1 dy \in \mathcal{C}_{2,2}^{1,2\beta}$ satisfying the algebraic relation :*

1. $\delta_2 G_2^{yy} = 0$
2. $\delta H_2^{yy} = G_2^{yy} \delta_2 y$
3. $\delta_1 \int_2 \int_2 d_2 y d_2 y = B_2^{yy} + \int_2 \int_2 dy d_2 y$
4. $\delta_2 B_2^{yy} = \delta_2 y \delta y$
5. $\delta_2 \int_2 d_2 y d_2 y = \delta_2 y \delta_2 y$
6. $\delta_2 \int_2 \int_2 dy d_2 y = \delta y \delta_2 y$
7. $\delta_1 y \int_2 \int_2 d_2 y d_2 y = \delta_1 y \circ_1 \int_2 \int_2 dy d_2 y - \int_2 \delta_2 y \circ_2 \delta y d_2 y + H_2^{yy}$
8. $\delta_1 y \delta_2 y = G_2^{yy} - \int_2 \int_2 dy d_2 y.$

Then for $\varphi \in C^5(\mathbb{R})$ the following equality holds :

$$\delta\varphi'(y) = \int_2 \varphi'(y) \int_1 dy + \int_2 \varphi''(y) \delta_1 y d_2 y + \varphi'(y)^{\sharp 1} \delta_2 y + r_1(y)$$

where

$$\begin{aligned} r_1(y) = & \varphi''(y)^{\sharp 1} \int_2 \int_2 d_2 y d_2 y + \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y) \\ & + \Lambda_2 [(\delta_2 \varphi'(y))^{\sharp 1} - \varphi''(y)^{\sharp 1} \delta_2 y - \varphi'''(y) \delta_1 y \circ_1 \delta y] \delta_2 y + \delta_2 \varphi''(y)^{\sharp 1} \int_2 \int_2 d_2 y d_2 y \\ & + \varphi'''(y) (\delta y \circ_1 \int_2 \int dy d_2 y) + \delta_2 \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y) \end{aligned}$$

Proof. By the one dimensional change of variable formula we have

$$\delta_2 \varphi(y) = \int_2 \varphi'(y) d_2 y = (1 - \Lambda_2 \delta_2) (\varphi(y) \delta_2 y + \varphi''(y) \int_2 \int_2 d_2 y d_2 y) \quad (66)$$

Then if we apply δ_1 to this equation we get

$$\begin{aligned} \delta\varphi(y) = & (1 - \Lambda_2 \delta_2) (\varphi'(y) \delta y + \varphi''(y) B_2^{yy} + \delta_1 \varphi'(y) \delta_2 y + \delta_1 \varphi''(y) \int_2 \int_2 d_2 y d_2 y + \varphi''(y) \int_2 \int dy d_2 y) \\ = & \int_2 \varphi'(y) \int_1 dy + (1 - \Lambda_2 \delta_2) (\delta_1 \varphi'(y) \delta_2 y + \delta_1 \varphi''(y) \int_2 \int_2 d_2 y d_2 y + \varphi''(y) \int_2 \int dy d_2 y) \end{aligned} \quad (67)$$

Expanding the two terms $\delta_1 \varphi'(y) \delta_2 y$, $\delta_1 \varphi''(y) \int_2 \int_2 d_2 y d_2 y$ and using the algebraic assumption gives

$$\begin{aligned} \delta_1 \varphi''(y) \delta_2 y + \varphi''(y) \int_2 \int_2 dy d_2 y + \delta_1 \varphi''(y) \int_2 dy d_2 y = & \varphi''(y) G_2^{yy} + \varphi'''(y) H_2^{yy} - \varphi'''(y) \int_2 \delta_2 y \circ_2 \delta y d_2 y \\ & + (\varphi'(y)^{\sharp 1} \delta_2 y + \varphi''(y)^{\sharp 1} \int_2 \int_2 d_2 y d_2 y + \\ & \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y)) \end{aligned} \quad (68)$$

Now if we combine the fact that $(1 - \Lambda_2 \delta_2) \int_2 \delta_2 y \circ_2 \delta y d_2 y = 0$ (ie: $\int_2 \delta_2 y \circ_2 \delta y d_2 y \in \mathcal{C}_{2,2}^{1,3\beta}$, $\delta_2 \int_2 \delta_2 y \circ_2 \delta y d_2 y \in \mathcal{C}_{2,3}^{1,3\beta}$) with equation (67) and (68) we obtain that

$$\begin{aligned} \delta\varphi(y) = & \int_2 \varphi'(y) \int_1 dy + \int_2 \varphi''(y) \delta_1 y d_2 y + (1 - \Lambda_2 \delta_2) (\varphi'(y)^{\sharp 1} \delta_2 y \\ & + \varphi''(y)^{\sharp 1} \int_2 \int_2 d_2 y d_2 y + \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y)). \end{aligned}$$

To obtain the needed result it suffice to expand the last term of this equality. □

Lemma 5.16. *Let $y \in \mathcal{CC}_{1,1}$ satisfying the assumption of lemma 5.15 and $\varphi \in C^5(\mathbb{R})$ then we have the formula :*

$$\begin{aligned} (\delta_2 \varphi'(y))^{\sharp 1} - \varphi''(y)^{\sharp 1} \delta_2 y - \varphi'''(y) \delta_1 y \circ_1 \delta y_{s_1 s_2 t_1 t_2} &= \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta y_{s_1 s_2 t_1 t_2})^2 \\ &+ \int_{[0,1]^2} k \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2}) dk dk' (2 \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} + (\delta y_{s_1 s_2 t_1 t_2})^2) \\ &+ \int_{[0,1]^3} \tilde{k} \varphi^{(v)}(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1} + k'' \delta_2 y_{s_1 t_1 t_2} + \bar{k} k'' \delta y_{s_1 s_2 t_1 t_2}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2})^2 dk dk' dk'' (\delta_1 y_{s_1 s_2 t_2})^2 \end{aligned} \quad (69)$$

where $\bar{k} = k k'$ and $\tilde{k} = k(1 - k'')$ in particular this give us that $r_1(y) \in \mathcal{CC}_{2,2}^{2\alpha, \beta}$.

Proof. By the usual Taylor formula we have that

$$\begin{aligned} \delta_2 \varphi'(y)_{s_1 s_2 t_1 t_1}^{\sharp 1} &= \int_{[0,1]^2} k (\varphi'''(y_{s_1 t_2} + \bar{k} \delta_1 y_{s_1 s_2 t_2}) - \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1})) dk dk' (\delta_1 y_{s_1 s_2 t_2})^2 \\ &+ 2 \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \\ &+ \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta y_{s_1 s_2 t_1 t_2})^2 \end{aligned} \quad (70)$$

Let denoted by $a_{s_1 s_2 t_1 t_2}$ the first term in the r.h.s of this equation. Then if we remark that $(\delta_1 y_{s_1 s_2 t_2})^2 = (\delta_1 y_{s_1 s_2 t_1})^2 + 2 \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_1} + (\delta y_{s_1 s_2 t_1 t_1})^2$ and use Taylor formula once again we obtain

$$\begin{aligned} a_{s_1 t_1 s_2 t_2} &= \int_{[0,1]^2} k \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk dk' \delta_2 y_{s_1 t_1 t_2} (\delta_1 y_{s_1 s_2 t_1})^2 \\ &+ \int_{[0,1]^2} k^2 k' \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta_1 y_{s_1 s_2 t_1})^2 \delta y_{s_1 s_2 t_1 t_2} \\ &+ \int_{[0,1]^3} \tilde{k} \varphi^{(v)}(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1} + k'' \delta_2 y_{s_1 t_1 t_2} + \bar{k} k'' \delta y_{s_1 s_2 t_1 t_2}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2})^2 dk dk' dk'' \\ &\times (\delta_1 y_{s_1 s_2 t_2})^2 \\ &+ \int_{[0,1]^2} k \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2}) dk dk' (2 \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} + (\delta y_{s_1 s_2 t_1 t_2})^2) \end{aligned} \quad (71)$$

The two last terms in the r.h.s of this equation lie in the space $\mathcal{CC}_{2,2}^{2\alpha, 2\beta}$ then we will focus on the two first denoted respectively by $a_{s_1 s_2 t_1 t_2}^1$ and $a_{s_1 s_2 t_1 t_2}^2$. By integration by part formula we get

$$\int_0^1 k^2 k' \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk \delta_1 y_{s_1 s_2 t_1} = \varphi'''(y_{s_1 t_1} + k' \delta_1 y_{s_1 s_2 t_1}) - 2 \int_0^1 k \varphi'''(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk.$$

Multiplying this equation by $\delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2}$ and integrating over k' give us

$$\begin{aligned} a_{s_1 s_2 t_1 t_2}^2 &= \int_0^1 \varphi'''(y_{s_1 t_1} + k' \delta_1 y_{s_1 s_2 t_1}) dk' \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \\ &- 2 \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \end{aligned} \quad (72)$$

on the other hand

$$\begin{aligned}
a_{s_1 s_2 t_1 t_2}^1 &= \int_{[0,1]^2} \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta_1 y_{s_1 s_2 t_1})^2 \delta_2 y_{s_2 t_1 t_2} \\
&\quad - \int_{[0,1]} \varphi'''(y_{s_1 t_1} + k \delta_1 y_{s_1 s_2 t_1}) dk \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \\
&= (\varphi''(y))^{\sharp 1} \delta_2 y_{s_1 s_2 t_1 t_2} - \int_{[0,1]} (\varphi'''(y_{s_1 t_1} + k \delta_1 y_{s_1 s_2 t_1}) - \varphi'''(y_{s_1 t_1})) dk \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2}.
\end{aligned} \tag{73}$$

If we combine equations (70) (71) (72) and (73) we obtain the needed result. \square

Now what we have in mind is to take $y = x^1$ in this last two lemma but for this we need to construct $\int_2 \int_2 d_2 x^1 d_2 x^1$ and $\int_2 \int dx^1 dx^1$ then as usual we must add some algebraic conditions

Hypothesis 5.17. Let $\alpha, \beta > 1/3$, $x \in \mathcal{C}\mathcal{C}_{1,1}$, $a = 1, 2$ and we assume the existence of $P_a^{xx} = \int_a \int_a d_a x d_a x \in \mathcal{C}\mathcal{C}_{1,2}^{*,2\beta}$ and $Q_a^{xx} = \int_a \int dx d_a x \in \mathcal{C}\mathcal{C}_{2,2}^{\alpha,2\beta}$ satisfying the algebraic relation

1. $\delta_a P_a^{xx} = \delta_a x \delta_a x$
2. $\delta_{\hat{a}} P_a^{xx} = B_a^{xx} + Q_a^{xx}$
3. $\delta_2 Q_a^{xx} = \delta x \delta_a x$
4. $\delta_{\hat{a}} x P_a^{xx} = \delta_{\hat{a}} x Q_a^{xx} + H_a^{xx} - \int_a \delta_a x \circ_a \delta x d_a x$
5. $\delta_{\hat{a}} x \delta_a x = G_a^{xx} - Q_a^{xx}$

where $\int_a \delta_a x \circ_a \delta x d_a x = \Lambda_a((\delta_a x \circ_a \delta x) \delta_a x + \delta_a x Q_a^{xx} + \delta x P_a^{xx} - \delta x \circ_a Q_a^{xx})$ and B_a^{xx} , H_a^{xx} are the iterated integrals given respectively in the hypothesis 5.1 and 5.11.

With this hypothesis we define

$$\left(\int_2 \int_2 d_2 x^1 d_2 x^1 \right)_{s_1 s_2 t_1 t_2} = (P_2^{xx})_{0t_1 t_2} + s((B_2^{xx})_{01t_1 t_2} + (Q_2^{xx})_{01t_1 t_2} + (s-1)(K_2^{xx})_{01t_1 t_2})$$

and

$$\left(\int_2 \int dx^1 d_2 x^1 \right)_{s_1 s_2 t_1 t_2} = (s_2 - s_1)((P_2^{xx})_{01t_1 t_2} + (s_2 - 1)(K_2^{xx})_{01t_1 t_2})$$

Now all the ingredients are ready to prove our main result

Theorem 5.18. Let $x \in \mathcal{C}\mathcal{C}_{1,1}^{\alpha,\beta}$ and $\varphi \in C^5(\mathbb{R})$ such that hypothesis 5.1, 5.8, 5.11 and 5.17 are satisfied then $\varphi(x) \in \mathcal{K}_x^{\alpha,\beta}$.

Proof. Let R^1 the remainder terms given by 65 then if we put $y = x^1$ in the lemma 5.15 we obtain

$$R^1 = \int_2 \varphi'(x^1) \int_1 dx^1 + \int_2 \varphi''(x^1) \delta_1 x^1 d_2 x^1 + \varphi'(x^1)^{\sharp 1} \delta_2 x^1 + r_1(x^1)$$

Now if we observe that:

$$(\varphi'(x^1) \delta_2 x^1)_{\square} = \left(\int_1 \varphi'(x^1) \int_2 dx^1 + \int_1 \varphi''(x^1) \delta_2 x^1 dx^1 \right)_{\square} - \nu_1(x)$$

where $\nu_1(x)$ is given in the lemma 5.14 we get that $R_\square^1 = r_1(x^1)_\square - \nu_1(x)_\square$ and $R_\square^2 = (r_2(x^2) - \nu_2(x))_\square$ then by the proposition we obtain that $R_\square = r_1(x^1)_\square - \nu_1(x)_\square + r_2(x^2)_\square - \nu_2(x)_\square - R_\square^{12}$ of course this relation give us the needed regularity of R on the unite square but if we take $X_{st} = x_{s_1+s(s_2-s_1);t_1+t(t_2-t_1)}$ for $(s, t) \in [0, 1]^2$ then is easy to see that X satisfy all the algebraic assumption and that $R_{s_1s_2t_1t_2}^x = R_\square^X$ which give us the result for any rectangle. \square

Now to simplify the notation we introduce the following definition.

Definition 5.19. Let $x \in \mathcal{C}_{1,1}$ a sheet satisfying the Hypothesis 5.1, 5.8, 5.11 and 5.17 then we denote by \mathbb{X} the collection of all iterated integrals giving in these Hypothesis and we call it Rough-Sheet associated to x and then we define $\mathcal{H}_{\alpha,\beta}$ the space which contain the rough sheet as the product of the Hölder space giving in these Hypothesis equipped with the product topology.

Now we have the following lemma:

Lemma 5.20. Let $\rho_1, \rho_2 \in (0, 1)$, x^1, x^2 two increments lying in $\mathcal{C}_{1,1}^{\rho_1, \rho_2}$, and $\varphi \in C^3(\mathbb{R})$. Then we have:

$$\|\delta_1 \varphi(x^1)\|_{\rho_1, 0} \lesssim \left(\sup_{s, t \in [0, 1]^2} |\varphi(x_{st}^1)| \right) \|\delta x^1\|_{\rho_1, 0} \quad (74)$$

and

$$\mathcal{N}_{\rho_1, \rho_2}(\varphi(x^1) - \varphi(x^2)) \lesssim c_{x^1, x^2} \mathcal{N}_{\rho_1, \rho_2}(x^1 - x^2) [1 + \mathcal{N}_{\rho_1, \rho_2}(x^1) + \mathcal{N}_{\rho_1, \rho_2}(x^2)]^2 \quad (75)$$

where we recall that $\mathcal{N}_{\rho_1, \rho_2}(\cdot)$ has been defined at equation (30). In the relation above we have also set

$$c_{x^1, x^2} = \sum_{i=1}^3 \sup_{(s, t) \in [0, 1]^2} |\varphi^{(i)}(x_{st}^1)| + \sup_{(s, t) \in [0, 1]^2} |\varphi^{(i)}(x_{st}^2)|$$

Using the concrete expression of the remainder term obtained previously, the continuity of the sewing map and this lemma we get easily the following continuity theorem.

Theorem 5.21. Let $x \in \mathcal{C}_{1,1}^{\alpha, \beta}$ and $\bar{x} \in \mathcal{C}_{1,1}^{\alpha, \beta}$ satisfying the hypothesis (5.1), (5.8), (5.11) and (5.17) and $\varphi \in C^8(\mathbb{R})$ then there exist a polynomial function $K \in C([0, +\infty[, [0, +\infty[)$ such that

$$\|\varphi(x)^\sharp - \varphi(\bar{x})^\sharp\|_{2\alpha, 2\beta} \lesssim_{\alpha, \beta} CK(\|\mathbb{X}\|_{\mathcal{H}_{\alpha, \beta}} + \|\bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}) \|\mathbb{X} - \bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}$$

and then

$$\left\| \iint \varphi(x) dx - \iint \varphi(\bar{x}) d\bar{x} \right\|_{\alpha, \beta} + \left\| \iint \varphi(x) d\omega(x) - \iint \varphi(\bar{x}) d\omega(\bar{x}) \right\|_{\alpha, \beta} \lesssim_{\alpha, \beta} CK(\|\mathbb{X}\|_{\mathcal{H}_{\alpha, \beta}} + \|\bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}) \times \|\mathbb{X} - \bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}$$

where $d\omega(x) := d_1 x d_2 x$, $\omega(\bar{x}) := d_1 \bar{x} d_2 \bar{x}$ and $C = \sum_{k \in \{1, \dots, 8\}} \|\varphi^{(k)}\|_{\infty, M}$ and $M = \|x\|_\infty + \|\bar{x}\|_\infty$

6 Enhancement of the fractional Brownian Sheet and Stratonovich formula

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, in this section we construct the rough-sheet associated to the fractional Brownian sheet x . Before staring with probabilistic computation let us recall the definition of such process.

Definition 6.1. *The process $(x_{st})_{(s,t) \in [0,1]^2}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called fractional Brownian sheet with hurst parameter $\alpha, \beta \in [0, 1]$ if x is a Gaussian process with covariance function*

$$R_{s_1 s_2 t_1 t_2} = 1/4(|s_1|^\alpha + |s_2|^\alpha + |s_2 - s_1|^\alpha)(|t_1|^\beta + |t_2|^\beta + |t_2 - t_1|^\beta). \quad (76)$$

With this definition we recall the following harmonisable representation for the fractional Brownian sheet

$$x_{st} \stackrel{law}{=} \int_{\mathbb{R}^2} \frac{e^{is\xi} - 1}{|\xi|^{\alpha+1/2}} \frac{e^{it\eta} - 1}{|\eta|^{\beta+1/2}} \hat{W}(d\xi, d\eta) \quad (77)$$

where \hat{W} is the Fourier transform of the white noise W (see [23]). Let us now state some extension of Garsia-Rodemich-Rumsey Lemma (see [4]) which will be useful to estimate the Hölder norm of random fields.

Lemma 6.2. *For $p > 1$ and $\alpha, \beta \in (\frac{1}{3}, \frac{1}{2}]$ there exist two non negative constants $C_1 = C_1(\alpha, \beta, p)$ and $C_2 = C_2(\alpha, p)$ such as for every $y \in \mathcal{C}\mathcal{C}_2$ and $R \in \mathcal{C}_2$ we have :*

$$\|\delta y\|_{\alpha, \beta} \leq C_1 U_{\alpha + \frac{2}{p}, \beta + \frac{2}{p}, p}^2(\delta y)$$

and

$$\|R\|_\alpha \leq C_2 (U_{\alpha + \frac{2}{p}, p}^1(R) + \|\delta^{1d} R\|_\alpha)$$

where

$$U_{\alpha_1, \alpha_2, \dots, \alpha_n, p}^n(V) := \left(\int_{[0,1]^{2n}} \frac{|V_{s_1^1 s_2^1 \dots s_1^n s_2^n}|^p}{\prod_{i=1}^n |s_2^i - s_1^i|^{\alpha_i}} ds_1^1 ds_2^1 \dots ds_1^n ds_2^n \right)^{\frac{1}{p}}$$

for $V \in \mathcal{C}_2^{\otimes n}$

Proof. Let $(s_1, s_2, t_1, t_2) \in [0, 1]^4$ then by the Garsia-Rodemich-Rumsey Lemma we have that :

$$|\delta y_{(s_1, s_2), (t_1, t_2)}|^p = |\delta_1 y_{(s_1, s_2)t_2} - \delta_1 y_{(s_1, s_2)t_1}|^p \lesssim_{\alpha, \beta, p} |t_2 - t_1|^{\beta p} \iint_{[0,1]^2} \frac{|\delta_1 y_{(s_1, s_2)v_2} - y_{(s_1, s_2)v_1}|^p}{|v_2 - v_1|^{\beta p + 2}} dv_1 dv_2$$

Now we remark that :

$$|\delta y_{(s_1, s_2), (t_1, t_2)}| = |\delta_1 y_{(s_1, s_2)t_2} - \delta_1 y_{(s_1, s_2)t_1}| = |\delta_2 y_{s_2(t_1, t_2)} - \delta_2 y_{s_1(t_1, t_2)}|$$

Then if we apply the Garsia-Rodemich-Rumsey again we obtain:

$$|\delta y_{(s_1, s_2), (t_1, t_2)}|^p \lesssim_{\alpha, \beta, p} |s_1 - s_2|^{\alpha p} |t_2 - t_1|^{\beta p} \iiint_{[0,1]^4} \frac{|\delta y_{(u_1, u_2)(v_1, v_2)}|^p}{|u_1 - u_2|^{\alpha p + 2} |v_2 - v_1|^{\beta p + 2}} du_1 du_2 dv_1 dv_2$$

For the proof of second inequality we refer the reader to [8]. □

Now our strategy is to regularize x in the following way: $\forall N \in \mathbb{N}$ we put

$$x_{st}^N := K_{\alpha,\beta} \iint_{\{|\xi|, |\eta| \leq N\}} \frac{e^{is\xi} - 1}{|\xi|^{\alpha+1/2}} \frac{e^{it\eta} - 1}{|\eta|^{\beta+1/2}} \hat{W}(d\xi, d\eta) \quad (78)$$

so we are able to define

$$\partial_1 \partial_2 x_{st}^N := K_{\alpha,\beta} \iint_{\{|\xi|, |\eta| \leq N\}} \frac{i\xi e^{is\xi}}{|\xi|^{\alpha+1/2}} \frac{i\eta e^{it\eta}}{|\eta|^{\beta+1/2}} \hat{W}(d\xi, d\eta)$$

And this allows us to construct rough sheet associated to x^N denoted in the following by \mathbb{X}^N . Now we will set out the main theorem of this section which will allow us to say that the fractional Brownian sheet can be enhanced in a rough sheet.

Theorem 6.3. *Let x^N the process given by the equation (78) and \mathbb{X}^N the associated rough sheet then there exists a random variable $\mathbb{X} \in \mathcal{H}_{h,h'}$ such that \mathbb{X}^N converges to \mathbb{X} in $L^p(\Omega, \mathcal{H}_{h,h'})$ for all $(h, h', p) \in (\frac{1}{3}, \alpha) \times (\frac{1}{3}, \beta) \times [1, +\infty)$.*

To prove the theorem 6.3 we will need the following lemma

Lemma 6.4. *Let $\alpha > 1/3$ and the function defined on \mathbb{R}^2 by :*

$$\mathcal{Q}(\xi, \eta) := \int_0^1 ds e^{is\xi} \int_0^s dv e^{iv\eta}$$

then \mathcal{Q} satisfies

1. $\mathcal{Q}(\xi, -\xi) = \frac{(1-\cos(\xi))+i(\xi-\sin(\xi))}{|\xi|^2}.$
2. $\mathcal{Q}(\xi_1, \xi_2) \lesssim \frac{1}{|x_i|}.$
3. $\mathcal{Q}(\xi_1, \xi_2) \lesssim \frac{1}{|x_1||x_2|} + \frac{1}{|x_1+x_2||x_i|}.$
4. $\mathcal{Q}(\xi_1, \xi_2) \lesssim 1.$
5. $\iint_{\mathbb{R}^2} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} d\xi d\eta < +\infty$

where $i \in \{1, 2\}$.

Proof. The properties 1,2,3,4 are easy to establish by a direct computation only the prove of the last assertion claim a bit more work. Indeed we begin by decomposing the plane in three region D, U and V given by :

1. $D = \left\{ (\xi, \eta) \in \mathbb{R}^2; |\xi + \eta| \leq \frac{\min(|\xi|, |\eta|)}{2} \right\}$
2. $U = \left\{ (\xi, \eta) \in \mathbb{R}^2; |\xi + \eta| \geq \frac{\max(|\xi|, |\eta|)}{2} \right\}$
3. $V = (D \cup U)^c$

Now when $(\xi, \eta) \in D$ we have that $2/3|\xi| \leq |\eta| \leq 3/2|\xi|$ which leads us by the third property given in Lemma to obtain the following bound :

$$|\mathcal{Q}(\xi, \eta)| \lesssim \frac{1}{|\xi||\xi + \eta|}$$

and then

$$\begin{aligned} \iint_{D \cap \{|\xi + \eta| \geq 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} d\xi d\eta &\lesssim \iint_{\{|\xi + \eta| \geq 1; |\xi| \geq 2|\xi + \eta|\}} \frac{1}{|\xi|^{4\alpha}|\xi + \eta|^2} d\xi d\eta \\ &\lesssim \int_1^{+\infty} dv \frac{1}{v^2} \int_{2v}^{+\infty} du \frac{1}{u^{4\alpha}} < +\infty \end{aligned}$$

Now if $|\xi + \eta| \leq 1$ we can estimate the integrand in the following way :

$$\frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} \lesssim \frac{1}{|\xi|^{4\alpha-2\gamma}}$$

where $\gamma \in [0, 1]$ and then we get :

$$\iint_{D \cap \{|\xi + \eta| \leq 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} d\xi d\eta \lesssim \int_0^1 dv \int_{2v}^{+\infty} du \frac{1}{u^{4\alpha-2\gamma}} < +\infty$$

as soon as $\gamma \in (2\alpha - 1, 2\alpha - 1/2)$ which shows that the integral is finite on D . Now on U and V we have that $|\xi|, |\eta| \lesssim |\xi + \eta|$ and hence we can estimate \mathcal{Q} by:

$$|\mathcal{Q}(\xi, \eta)| \lesssim \frac{1}{|\xi||\eta|}$$

then we obtain

$$\iint_{(U \cup V) \cap \{|\xi| > 1, |\eta| > 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} d\xi d\eta \lesssim \iint_{[1, +\infty)^2} |\xi\eta|^{-4\alpha-1} d\xi d\eta < +\infty$$

In the region of U and V where $|\xi|, |\eta| \leq 1$ we bound the integrand in the following manner.

$$\frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} \lesssim |\xi\eta|^{1-2\alpha}$$

then

$$\iint_{(U \cup V) \cap \{|\xi| \leq 1, |\eta| \leq 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} d\xi d\eta \lesssim \left(\int_{\{|\xi| \leq 1\}} |\xi|^{1-2\alpha} d\xi \right)^2$$

The same bound for the region $(U \cup V) \cap \{|y| \leq 1, |x| \geq 1\}$ combined with the fact that $\mathcal{Q}(\xi, \eta) \lesssim \min(|\xi|^{-1}, |\eta|^{-1})$ gives

$$\frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1}|\eta|^{2\alpha-1}} \lesssim |\xi|^{-1-2\alpha}|\eta|^{-2\alpha+1}$$

This shows that our kernel is integrable on $(U \cup V) \cap \{|\xi| \leq 1, |\eta| \geq 1\}$ and by symmetry we obtain the integrability in the remaining area which completes the proof. \square

6.1 Proof of theorem (6.3)

Proof. We will decompose the proof of the theorem in two step. In a first step we give the bound for the rough sheet \mathbb{X}^N in $L^2(\Omega)$ for fixed parameters and in the second step we will use a variant of Garsia-Rodemich-Rumsey inequality to prove that the sequence $(\mathbb{X}^N)_N$ is a Cauchy sequence in $\mathbb{L}^p(\Omega, \mathcal{H}_{h,h'})$.

Step 1: Estimation. Let $A^{NMx} := A^{Nx} - A^{Mx} = \delta(x^N - x^M)$ for $M \leq N$ and similar notation for all other terms of the rough sheet. Now is not difficult to see that

$$\begin{aligned} \mathbb{E}[|A_{(s_1, s_2)(t_1, t_2)}^{NMx}|^2] &= \int_{\{||(\xi, \eta)||_\infty \in [M, N]\}} \frac{|e^{is_2\xi} - e^{is_1\xi}|^2}{|\xi|^{1+2\alpha}} \frac{|e^{it_2\eta} - e^{it_1\eta}|^2}{|\eta|^{1+2\beta}} d\xi d\eta \\ &\lesssim (s_2 - s_1)^{2\alpha} (t_2 - t_1)^{2\beta} (I_A^M)_{(s_1, s_2)(t_1, t_2)} \end{aligned}$$

where

$$(I_A^M)_{(s_1, s_2)(t_1, t_2)} = \int_{\{||((t_2 - t_1)x, (s_2 - s_1)y)||_\infty \geq M(s_2 - s_1)(t_2 - t_1)\}} |x|^{-1-2\alpha} |e^{ix} - 1|^2 |y|^{-1-2\beta} |e^{iy} - 1|^2 dx dy$$

And let us remark that for $(\alpha, \beta) \in (1/3, 1/2]$ we have :

$$\int_{\mathbb{R}^2} |\xi|^{-1-2\alpha} |e^{i\xi} - 1|^2 |\eta|^{-1-2\beta} |e^{i\eta} - 1|^2 d\xi d\eta < +\infty$$

these imply

$$\sup_{M \in \mathbb{N}, (s_1, s_2, t_1, t_2) \in [0, 1]^4} |(I_A^M)_{(s_1, s_2)(t_1, t_2)}| < +\infty$$

and

$$\lim_{M \rightarrow \infty} (I_A^M)_{(s_1, s_2)(t_1, t_2)} = 0$$

for $s_1 \neq s_2$ and $t_1 \neq t_2$. Now in what follows we prove similar bound for the other component of the rough sheet. By Wick theorem

$$\begin{aligned} \mathbb{E}[|C_{(s_1, s_2)(t_1, t_2)}^{Nxx}|^2] &= \left| \int_{\mathbb{R}^2} (K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, -\xi) d\xi \right|^2 + \int_{\mathbb{R}^4} |(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta)|^2 d\xi d\eta \\ &\quad + \left| \int_{\mathbb{R}^4} (K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta) \overline{(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\eta, \xi)} d\xi d\eta \right| \end{aligned} \quad (79)$$

where

$$(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta) := \frac{i\xi i\eta}{|\xi|^{\alpha+1/2} |\eta|^{\alpha+1/2}} \mathcal{Q}_{s_1 s_2}(\xi, \eta) \frac{i\xi' i\eta'}{|\xi'|^{\beta+1/2} |\eta'|^{\beta+1/2}} \mathcal{Q}_{t_1 t_2}(\xi', \eta') \chi_{\{||(\xi, \xi', \eta, \eta')||_\infty \in [M, N]\}}$$

for $\xi := (\xi, \xi') \in \mathbb{R}^2$ et $\eta = (\eta, \eta') \in \mathbb{R}^2$ and $\mathcal{Q}_{s_1 s_2}(\xi, \eta) := e^{is_1(\xi+\eta)} (s_2 - s_1)^2 \mathcal{Q}((s_2 - s_1)\xi, (s_2 - s_1)\eta)$ with \mathcal{Q} is the function defined in the Lemma (6.4), which gives us by change of variable formula that

$$\int_{\mathbb{R}^4} |(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta)|^2 d\xi d\eta \lesssim (s_2 - s_1)^{4\alpha} (t_2 - t_1)^{4\beta} (I_C^{1,M})_{(s_1, s_2)(t_1, t_2)}$$

where

$$(I_C^{1,M})_{(s_1,s_2)(t_1,t_2)} := \int_{\mathbb{R}^4} \chi_{\{|| (s_2-s_1)\xi, (s_2-s_1)\eta, (t_2-t_1)\xi', (t_2-t_1)\eta' ||_\infty \geq M\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi\eta|^{2\alpha-1}} \frac{|\mathcal{Q}(\xi', \eta')|^2}{|\xi'\eta'|^{2\beta-1}} d\xi d\eta$$

Now the Lemma 6.4 gives :

$$\int_{\mathbb{R}^4} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi\eta|^{2\alpha-1}} \frac{|\mathcal{Q}(\xi', \eta')|^2}{|\xi'\eta'|^{2\beta-1}} d\xi d\eta < +\infty$$

and then

$$\sup_{M \in \mathbb{N}, (s_1, s_2, t_1, t_2) \in [0,1]^4} (I_C^{1,M})_{(s_1,s_2)(t_1,t_2)} < +\infty$$

and

$$\lim_{M \rightarrow \infty} (I_C^{1,M})_{(s_1,s_2)(t_1,t_2)} = 0$$

for $s_2 \neq s_1$ and $t_2 \neq t_1$ and by Cauchy-Schwartz inequality we have :

$$|\int_{\mathbb{R}^4} (K_{(s_1,s_2)(t_1,t_2)}^{NM})(\xi, \eta) \overline{(K_{(s_1,s_2)(t_1,t_2)}^{NM})(\eta, \xi)} d\xi d\eta| \leq \int_{\mathbb{R}^4} |(K_{(s_1,s_2)(t_1,t_2)}^{NM})(\xi, \eta)|^2 d\xi d\eta$$

Then is remind to bound the first term appearing in the sum of the equation (79), indeed

$$|\int_{\mathbb{R}^2} (K_{(s_1,s_2)(t_1,t_2)}^{NM})(\xi, -\xi) d\xi|^2 \lesssim (s_2 - s_1)^{2\alpha} (t_2 - t_1)^{2\beta} (I_C^{2,M})_{(s_1,s_2)(t_1,t_2)}$$

where

$$(I_C^{2,M})_{(s_1,s_2)(t_1,t_2)} = \int_{\mathbb{R}^2} \chi_{\{|| (s_2-s_1)\xi, (t_2-t_1)\xi' ||_\infty \geq M\}} |\xi|^{-1-2\alpha} |\xi'|^{1-2\beta} (1 - \cos(\xi))(1 - \cos(\xi')) d\xi d\xi'$$

then is no difficult to see that $I_C^{2,M}$ satisfies the same property that $I_C^{1,M}$. Now if we put $I_C^M := I_C^{2,M} + I_C^{1,M}$ is easy to see that :

$$\mathbb{E}[|C_{(s_1,s_2)(t_1,t_2)}^{NMxx}|^2] \lesssim (s_2 - s_1)^{2\alpha} (t_2 - t_1)^{2\beta} (I_C^M)_{(s_1,s_2)(t_1,t_2)}$$

when (I_C^M) satisfy :

$$\sup_{M \in \mathbb{N}, (s_1, s_2, t_1, t_2) \in [0,1]^4} (I_C^M)_{(s_1,s_2)(t_1,t_2)} < +\infty$$

$$\lim_{M \rightarrow \infty} (I_C^M)_{(s_1,s_2)(t_1,t_2)} = 0$$

for $s_2 \neq s_1$ and $t_2 \neq t_1$. All the other term of \mathbb{X}^N can be estimate by the same argument and satisfy the same type of bound to see these let us treat a more complex term of the rough sheet.

$$\mathbb{E}[|(E_1^{NMxxx})_{(s_1,s_2,s_3,s_3)(t_1,t_2)}|^2] = \sum_{l=1}^{15} \mathcal{J}_l$$

where $(\mathcal{J}_l)_l$ is the different Wick contraction given by:

$$1. \mathcal{J}_1 = \int_{\mathbb{R}^6} |(G_{(s_1,s_2,s_3,s_4)(t_1,t_2)}^{NM})(a, b, c)|^2 da db dc$$

2. $\mathcal{J}_2 = \int_{\mathbb{R}^2} |\int_{\mathbb{R}^2} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -a) da|^2 db$
3. $\mathcal{J}_3 = \int_{\mathbb{R}^2} |\int_{\mathbb{R}^2} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, -a, b) da|^2 db$
4. $\mathcal{J}_4 = \int_{\mathbb{R}^2} |\int_{\mathbb{R}^2} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -b) db|^2 da$
5. $\mathcal{J}_5 = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -a) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(b, c, -c)} dadbdc$
6. $\mathcal{J}_6 = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -a) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(c, -c, b)} dadbdc$
7. $\mathcal{J}_7 = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, -a, b) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, c, -c)} dadbdc$
8. $\mathcal{J}_8 = \overline{\mathcal{J}_5}, \mathcal{J}_9 = \overline{\mathcal{J}_6}, \mathcal{J}_{10} = \overline{\mathcal{J}_7}$
9. $\mathcal{J}_k = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, c) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(\sigma_k(a), \sigma_k(b), \sigma_k(c))} dadbdc$

where σ_k , $k \in \{11, 12, 13, 14, 15\}$ is a permutation of three elements different of identity and for $a = (a, a'), b = (b, b'), c = (c, c') \in \mathbb{R}^2$ the kernel is given by :

$$(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, c) := \left(\frac{iaib}{|a|^{\alpha+1/2}|b|^{\alpha+1/2}} \mathcal{Q}_{s_1 s_2}(a, b) \right) \left(\frac{ia'ic'}{|b'|^{\beta+1/2}|z'|^{\beta+1/2}} \mathcal{Q}_{t_1 t_2}(b', c') \right) \\ \times \left(\frac{e^{is_4 c} - e^{is_3 c}}{|c|^{\alpha+1/2}} \right) \left(\frac{e^{it_1 a'} - 1}{|a'|^{\beta+1/2}} \right) \chi_{\{||(a, a', b, b', c, c')||_{\infty} \in [M, N]\}}$$

The study of theses integral is confined to study the first four term in fact if $k \in \{11, 12, 13, 14, 15\}$ then by Cauchy-Schwartz we have $|\mathcal{J}_k| \leq \mathcal{J}_1$, $|\mathcal{J}_5|^2 \leq \mathcal{J}_2 \mathcal{J}_4$, $|\mathcal{J}_6|^2 \leq \mathcal{J}_2 \mathcal{J}_3$ and $|\mathcal{J}_7|^2 \leq \mathcal{J}_3 \mathcal{J}_4$. By variable change formula we have that

$$|\mathcal{J}_1| \lesssim (s_2 - s_1)^{4\alpha} (s_4 - s_3)^{2\alpha} (t_2 - t_1)^{4\beta} t_1^{2\beta} (I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)}$$

where

$$(I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)} := \int_{\mathbb{R}^6} \chi_{\mathcal{D}_{s_1 s_2 s_3 s_4 t_1 t_2}^M} \frac{|\mathcal{Q}(a, b)|^2}{|ab|^{2\alpha-1}} \frac{|\mathcal{Q}(b', c')|^2}{|a'b'|^{2\beta-1}} \frac{|e^{ic} - 1|^2}{|c|^{2\alpha+1}} \frac{|e^{ic} - 1|^2}{|c|^{2\beta+1}} dadbdc$$

where \mathcal{X} is the Indicator function and $\mathcal{D}_{s_1 s_2 s_3 s_4 t_1 t_2}^M$ is decreasing to the empty set when M tend to infinity for $s_1 \neq s_2, t_1 \neq t_2$ and $s_3 \neq s_4$ these statement with the fact that :

$$\int_{\mathbb{R}^6} \frac{|\mathcal{Q}(a, b)|^2}{|ab|^{2\alpha-1}} \frac{|\mathcal{Q}(b', c')|^2}{|a'b'|^{2\beta-1}} \frac{|e^{ic} - 1|^2}{|c|^{2\alpha-1}} \frac{|e^{iz} - 1|^2}{|z|^{2\beta-1}} dadbdc < +\infty$$

give us that

$$\sup_{M \in \mathbb{N}, (s_1, s_2, s_3, s_4, t_1, t_2) \in [0, 1]^6} (I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)} < +\infty$$

and

$$\lim_{M \rightarrow +\infty} (I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)} = 0$$

Now by Cauchy-Schwartz we have :

$$\mathcal{J}_2 \lesssim \left(\int_{\mathbb{R}^4} |(K_{(s_1, s_2)}^{NM}(t_1, t_2))|(a, b)|^2 da db \right) \int_{\mathbb{R}} |a|^{-1-2\alpha} |e^{-is_4 a} - e^{-is_3 a}|^2 da \\ \left(\int_{\mathbb{R}} |a'|^{-1-2\beta} |e^{it_1 a'} - 1|^2 da' \right)$$

all these integrals have already studied and then we obtain the good estimate for \mathcal{J}_2 . The estimation of \mathcal{J}_3 and \mathcal{J}_4 is obtained by the same technique.

Step 2: Convergence of \mathbb{X}^N . We prove some Garsia-Rodemich-Rumsey inequalities for our objects and then we obtain the convergence of the sheet X^N . To give an idea we will first show the convergence of our first term in fact by the Lemma 6.2 and the Gaussian hypercontractivity we have that :

$$\mathbb{E}[|A^{NMx}|_{h, h'}^p] \lesssim_{h, h', p, \alpha, \beta} \iint_{[0, 1]^4} \frac{\mathbb{E}[|A_{s_1 s_2 t_1 t_2}^{NMx}|^p]}{|s_2 - s_1|^{hp+2} |t_2 - t_1|^{h'p+2}} ds_1 ds_2 dt_1 dt_2 \\ \lesssim \iint_{[0, 1]^4} |s_2 - s_1|^{(\alpha-h)p-2} |t_2 - t_1|^{(\beta-h')p-2} ((I_A^M)_{(s_1, s_2)(t_1, t_2)})^{\frac{p}{2}} ds_1 ds_2 dt_1 dt_2$$

where $h < \alpha$ and $h' < \beta$. Then if p is large enough these last integral go to zero when M go to infinity by dominate convergence and this give us the convergence of A^{NMx} . Now we will use the same argument for the other terms and for this we must establish the following estimate :

Lemma 6.5. *let z and y two smooth sheets, $(h, h') \in (0, \infty)^2$ and $p > 1$ then the following inequalities :*

$$\|B_1^{zz} - B_1^{yy}\|_{2h, h'}^p \lesssim_{h, h', p} (U_{2h, h', p}^2 (B_1^{zz} - B_1^{yy}))^p + (U_{2h, h', h'}^3 (D_2^{zz} - D_2^{yy}))^p + \|\delta_1 z \delta z - \delta_1 y \delta y\|_{2h, h'}^p \\ + \|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}})^p \\ \|C^{zz} - C^{yy}\|_{2h, 2h'}^p \lesssim_{h, h', p} (U_{2h, 2h', p}^2 (C^{zz} - C^{yy}))^p + (U_{2h+\frac{2}{p}, h', h', p}^3 (D_2^{zz} - D_2^{yy}))^p \\ + (U_{h, h, 2h'+\frac{2}{p}, p}^3 (D_1^{zz} - D_1^{yy}))^p + (\|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}} (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}))^p \\ \|D_1^{zz} - D_1^{yy}\|_{2h, 2h'}^p \lesssim_{h, h', p} (U_{h, h, 2h'}^3 (D_1^{zz} - D_1^{yy}))^p + (\|\delta z - \delta y\|_{h+\frac{1}{p}, h'} (\|\delta z\|_{h+\frac{1}{p}, h'} + \|\delta y\|_{h+\frac{1}{p}, h'}))^p$$

hold.

Proof. Let us prove the three first inequalities because all the others are obtained by the same techniques. Lemma 6.2 applied in the first direction and using the fact that $\delta_1 B_1^{zz} = \delta_1 z \delta z$ we obtain that

$$|(B_1^{zz} - B_1^{yy})_{s_1 s_2 t_1 t_2}|^p \lesssim_{h, h', p} |s_2 - s_1|^{2hp} \left(\int_{[0, 1]^2} \frac{|(B_1^{zz} - B_1^{yy})_{u_1 u_2 t_1 t_2}|^p}{|u_2 - u_1|^{2hp+2}} du_1 du_2 \right. \\ \left. + \|(\delta_1 z \delta z - \delta_1 y \delta y)_{t_1 t_2}\|_{2h}^p \right) \quad (80)$$

Now we have to deal with the other direction. The second term appearing in the right side of this inequality can be estimated by $|t_2 - t_1|^{h'p} \|\delta_1 x \delta x - \delta_1 y \delta y\|_{2h, h'}^p$ for the first term we need to apply

Lemma 6.2 once again

$$\begin{aligned} \int_{[0,1]^2} \frac{|(B_1^{zz} - B_1^{yy})_{u_1 u_2 t_1 t_2}|^p}{|u_2 - u_1|^{2hp+2}} du_1 du_2 &\lesssim_{h,h',p} |t_2 - t_1|^{h'p} ((U_{2h,h',p}^2 (B_1^{zz} - B_1^{yy}))^p \\ &+ \int_{[0,1]^2} \frac{\|\delta_2(B_1^{zz} - B_1^{yy})_{u_1 u_2}\|_{h'}^p}{|u_2 - u_1|^{hp+2}} du_1 du_2) \end{aligned} \quad (81)$$

then it suffices to note that

$$\begin{aligned} \|\delta_2(B_1^{zz} - B_1^{yy})_{u_1 u_2}\|_{h'}^p &= \|(\mu_2 D_2^{zz} - \mu_2 D_2^{yy})_{u_1 u_2}\|_{h'}^p \\ &\leq \|(D_2^{zz} - D_2^{yy})_{u_1 u_2}\|_{\mathcal{C}_2^{h'} \otimes \mathcal{C}_2^{h'}}^p \\ &\lesssim_{h,h',p} \int_{[0,1]^4} \frac{|(D_2^{zz} - D_2^{yy})_{u_1 u_2 v_1 v_2 v_3 v_4}|^p}{|v_4 - v_3|^{h'p+2} |v_2 - v_1|^{h'p+2}} dv_1 dv_2 dv_3 dv_4 \\ &\lesssim_{h,h',p} |u_2 - u_1|^{hp+2} ((U_{h+\frac{2}{p},h',h',p}^3 (D_2^{zz} - D_2^{yy}))^p \\ &+ \int_{[0,1]^4} \frac{\|(\delta z \otimes_2 \delta z - \delta y \otimes_2 \delta y)_{v_1 v_2 v_3 v_4}\|_{h+\frac{2}{p}}^p}{|v_4 - v_3|^{h'p+2} |v_2 - v_1|^{h'p+2}} dv_1 dv_2 dv_3 dv_4). \end{aligned} \quad (82)$$

Now the last term in the right side of this inequality can be bounded by $\|\delta z - \delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p},h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}})^p$ then putting all these bound together gives

$$\begin{aligned} \|B_1^{zz} - B_1^{yy}\|_{2h,h'}^p &\lesssim_{h,h',p} (U_{2h,h',p}^2 (B_1^{zz} - B_1^{yy}))^p + (U_{2h,h',h'}^3 (D_2^{zz} - D_2^{yy}))^p + \|\delta_1 z \delta z - \delta_1 y \delta y\|_{2h,h'}^p \\ &+ \|\delta z - \delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p},h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}})^p \end{aligned}$$

then we have prove the first inequality. Now by the Lemma 6.2 once again we have that

$$|C_{s_1 s_2 t_1 t_2}^{zz} - C_{s_1 s_2 t_1 t_2}^{yy}| \lesssim_{h,h',p} |s_2 - s_1|^{2hp} ((U_{2h,p}^1 (C_{t_1 t_2}^{zz} - C_{t_1 t_2}^{yy}))^p + \|\delta_1 (C^{zz} - C^{yy})_{t_1 t_2}\|_{2h}^p)$$

with $\delta_1 C_{zz} = \mu_1 D_1^{zz}$ then by same argument has before we have

$$\begin{aligned} \|\delta_1 (C^{zz} - C^{yy})_{t_1 t_2}\|_{2h}^p &\lesssim_{h,h',p} |t_2 - t_1|^{2h'p} ((U_{h,h,2h',p}^3 (D_1^{zz} - D_1^{yy}))^p \\ &+ \|\delta z - \delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p},h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}})^p) \end{aligned} \quad (83)$$

and

$$\begin{aligned} (U_{2h,p}^1 (C_{t_1 t_2}^{zz} - C_{t_1 t_2}^{yy}))^p &\lesssim_{h,h',p} |t_2 - t_1|^{2h'p} ((U_{2h,2h',p}^2 (C^{zz} - C^{yy}))^p \\ &+ \int_{[0,1]^2} \frac{\|\delta_2 (C^{zz} - C^{yy})_{u_1 u_2}\|_{2h'}^p}{|u_2 - u_1|^{hp+2}} du_1 du_2) \\ &\lesssim_{h,h',p} |t_2 - t_1|^{2h'p} ((U_{2h,2h',p}^2 (C^{zz} - C^{yy}))^p + (U_{2h+\frac{2}{p},h',h',p}^3 (D_2^{zz} - D_2^{zz}))^p \\ &+ \|\delta z - \delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p},h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p},h'+\frac{2}{p}})^p) \end{aligned} \quad (84)$$

Putting these two bound together we obtain the second inequality. This concludes the proof of the Lemma \square

To obtain convergence of all the term of the rough sheet it suffices to take $x = X^N$ and $y = X^M$ in the lemma (6.5). Let us give an example :

$$\begin{aligned} \mathbb{E}[||C^{NMxx}||_{2h,2h'}^p] &\lesssim_{h,h',p} \int_{[0,1]^4} |u_2 - u_1|^{2(\alpha-h)p-2} |v_2 - v_1|^{2(\beta-h')p-2} ((I_C^M)_{(u_1,u_2)(v_1,v_2)})^{\frac{p}{2}} du_1 du_2 dv_1 dv_2 \\ &+ \int_{[0,1]^6} |u_2 - u_1|^{2(\alpha-h)p-4} |v_2 - v_1|^{(\beta-h')p-2} |v_4 - v_3|^{(\beta-h')p-2} ((I_{D_2}^M)_{(u_1,u_2)(v_1,v_2,v_3,v_4)})^{\frac{p}{2}} du_1 du_2 dv_1 dv_2 dv_3 dv_4 \\ &+ \int_{[0,1]^6} |v_2 - v_1|^{2(\beta-h')p-4} |u_2 - u_1|^{(\alpha-h')p-2} |u_4 - u_3|^{(\alpha-h')p-2} ((I_{D_1}^M)_{(u_1,u_2,u_3,u_4)(v_1,v_2)})^{\frac{p}{2}} dv_1 dv_2 du_1 du_2 du_3 du_4 \\ &+ \mathbb{E}[||A^{NMx}||_{h+2/p,h'+2/p}^{2p}]^{1/2} \mathbb{E}[(||\delta x^N||_{h+2/p,h'+2/p} + ||\delta x^M||_{h+2/p,h'+2/p})^{2p}]^{1/2} \end{aligned}$$

where we used that our term is in the second chaos of \hat{W} and the Gaussian hypercontractivity (see for example [17]). Then if p is large enough the three first terms of right side go to zero when $M \rightarrow \infty$ by dominate convergence and the last term was already studied this allow us to conclude that C^{NMxx} is a Cauchy sequence in $L^p(\Omega, \mathcal{C}_{2,2}^{2h,2h'})$. The other terms can be treated similarly. \square

This construction and theorem (5.6) allows us to define the two integrals

$$\iint f'(x) dx, \quad \iint f''(x) d_1 x d_2 x$$

for a function $f \in C^{10}(\mathbb{R})$. Our goal is then to use the continuity result (5.21) to obtain the Stratonovich change of variable formula, for that we need an another assumption on the function f which allow us to control the constant C appearing in (5.21).

Definition 6.6. Let $k \in \mathbb{N}$, we will say that a function $f \in C^k(\mathbb{R})$ satisfies the growth condition (GC) if there exist positive constants c and λ such that

$$\lambda < \frac{1}{4 \max_{s,t \in [0,1]} (R_s^1 R_t^2)}, \quad \text{and} \quad \max_{l=0,\dots,k} |f^{(l)}(\xi)| \leq c e^{\lambda |\xi|^2} \quad \text{for all } \xi \in \mathbb{R}. \quad (85)$$

And a preliminary result ensures that x^N satisfy some uniform exponential integrability.

Proposition 6.7. There exist $\lambda > 0$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[e^{\lambda \sup_{(s,t) \in [0,1]^2} |x_{s;t}^N|^2} \right] < +\infty. \quad (86)$$

Proof. See [3]. \square

Now is not difficult to obtain the following result

Theorem 6.8. Let $f \in C^{10}(\mathbb{R})$ function satisfying the (GC) with a small parameter $\lambda > 0$ then

$$\delta f(x) = \iint f'(x) dx + \iint f''(x) d_1 x d_2 x \quad (87)$$

moreover the following convergence holds

$$\iint f'(x^N) dx^N \xrightarrow{N \rightarrow +\infty} \iint f'(x) dx, \quad \iint f''(x^N) d_1 x^N d_2 x^N \xrightarrow{N \rightarrow +\infty} \iint f''(x) d_1 x d_2 x \quad (88)$$

in $L^p(\Omega, \mathcal{C}_{2,2}^{\alpha-\varepsilon, \beta-\varepsilon})$ for $\varepsilon > 0$ and $0 < \lambda < \lambda(p)$.

Proof. Due to Theorem (5.21) we have that

$$\left\| \iint f'(x^N) dx^N - \iint f'(x) dx \right\|_{\alpha-\varepsilon, \beta-\varepsilon} \lesssim C^N K(\|\mathbb{X}^N\| + \|\mathbb{X}\|) \|\mathbb{X}^N - \mathbb{X}\|_{\mathcal{H}_{\alpha-\varepsilon, \beta-\varepsilon}}$$

Then using the Hölder inequality we can see that the needed convergence is only due to the fact that

$$\mathbb{E}[\|\mathbb{X}^N - \mathbb{X}\|_{\alpha-\varepsilon, \beta-\varepsilon}^a] \rightarrow^{N \rightarrow +\infty} 0, \quad \sup_N \mathbb{E}[K(\|\mathbb{X}^N\| + \|\mathbb{X}\|)^b] < +\infty, \quad \sup_N \mathbb{E}[(C^N)^c] < \infty$$

for some $a, b, c > 0$, Now the two first affirmation are given by the theorem (6.3) and for the third it suffices to recall that

$$C = \sum_{i=1}^{10} \sup_{|\xi| \leq \|x\|_\infty + \|x^N\|_\infty} |f^{(i)}(\xi)| \lesssim e^{\lambda \|x\|_\infty^2} e^{\lambda \|x^N\|_\infty^2}$$

where we have used that f satisfy the (GC) and then using Hölder inequality, Fernique's Theorem and (86) we can see that $\sup_N \mathbb{E}[(C^N)^c] < +\infty$ which gives the convergence (88). Now to obtain the formula (87) it suffices to use the fact that

$$\mathcal{N}_{\alpha-\varepsilon, \beta-\varepsilon}(f(x^N) - f(x)) \rightarrow^{N \rightarrow +\infty} 0$$

in $L^p(\Omega)$ for $p > 1$ due to the Lemma (5.20) and the fact that $\sup_N \mathbb{E}[(C^N)^p] < +\infty$, and this end the proof. \square

6.2 The Brownian case

In [2] the authors give a definition a la Itô for the multidimensional integral in the case of the Brownian sheet, the aim of this section is to compare their notion of integration with the integral that we defined. More precisely we will show that the two concepts coincide when the rough sheet is understood a la Itô. In all this subsection x is a Brownian sheet and $\mathbb{X}^{It\hat{o}}$ the rough sheet be associated where all the iterated integrals are understood in the sense given in [2].

Of course using the same argument as in the lemma 6.5 we can see that our objects satisfy the regularity expected. For example

$$\begin{aligned} \mathbb{E}[\|B^{It\hat{o}xx}\|_{2h, h'}^p] &\lesssim_{h, h', p} \int_{[0,1]^4} \frac{\mathbb{E}[|(B_1^{It\hat{o}xx})_{u_1 u_2 v_1 v_2}|^p]}{|u_2 - u_1|^{2hp+2} |v_2 - v_1|^{h'p+2}} du_1 du_2 dv_1 dv_2 \\ &+ \int_{[0,1]^6} \frac{\mathbb{E}[|(D_2^{It\hat{o}xx})_{u_1 u_2 v_1 v_2 v_3 v_4}|^p]}{|u_2 - u_1|^{2hp+2} |v_4 - v_3|^{h'p+2} |v_2 - v_1|^{h'p+2}} du_1 du_2 du_3 du_4 dv_1 dv_2 \\ &+ \mathbb{E}[\|\delta_1 x \delta x\|_{2h, h'}^p] + \mathbb{E}[\|\delta x\|_{h+2/p, h'+2/p}^{2p}] \end{aligned} \quad (89)$$

but by a simple computation we have that

$$\mathbb{E}[|(B_1^{It\hat{o}xx})_{u_1 u_2 v_1 v_2}|^p] = c_p^1 u_1^{p/2} (u_2 - u_1)^p (v_2 - v_1)^{p/2}$$

and

$$\mathbb{E}[|(D_2^{It\hat{o}xx})_{u_1 u_2 v_1 v_2 v_3 v_4}|^p] = c_p^2 (u_2 - u_1)^p (v_2 - v_1)^{p/2} (v_4 - v_3)^{p/2}$$

then if $h, h' < 1/2$ and p large enough then the r.h.s of the equation is finite and this gives that $B_1^{It\hat{o}XX} \in \mathcal{C}_{2,2}^{2h, h'}$. To compare the two definition of integration we will show first that the two definition of boundary integral coincide.

Proposition 6.9. Let $\varphi \in C^2(\mathbb{R})$ satisfying the (GC) then the integral

$$I_{s_1 s_2 t_1 t_2}^{It\hat{o}, b_1} = \int_{s_1}^{s_2} \varphi(x_{st_1}) \int_{t_1}^{t_2} \hat{d}_{st} x_{st}$$

with \hat{d} is the $It\hat{o}$ differential admit a continuous version which coincide with

$$I_{s_1 s_2 t_1 t_2}^{Rough, b_1} = \int_{s_1}^{s_2} \varphi(x_{st_1}) \int_{t_1}^{t_2} d_{st} x_{st}$$

where I^{rough, X, b_1} are given by the Proposition 5.4.

Proof. Let $\pi = (s_i)_i$ a dissection of the interval $[s_1, s_2]$. Now by definition we have that

$$I_{s_1 s_2 t_1 t_2}^{It\hat{o}x, b_1} = \mathbb{P} - \lim_{|\pi| \rightarrow 0} \sum_i \varphi(x_{s_i t_1}) \delta x_{s_i s_{i+1} t_1 t_2}$$

and

$$I_{s_1 s_2 t_1 t_2}^{Rough, x, b_1} = a.s - \lim_{|\pi| \rightarrow 0} \sum_i \varphi(x_{s_i t_1}) \delta x_{s_i s_{i+1} t_1 t_2} + \varphi'(x_{s_i t_1}) (B_1^{It\hat{o}, xx})_{s_i s_{i+1} t_1 t_2}$$

but

$$\begin{aligned} \mathbb{E} \left[\left| \sum_i \varphi'(x_{s_i t_1}) (B_1^{It\hat{o}xx})_{s_i s_{i+1} t_1 t_2} \right|^2 \right] &= \sum_i \mathbb{E}[\varphi'(x_{s_i t_1})^2] \mathbb{E}[(B_1^{It\hat{o}, xx})_{s_i s_{i+1} t_1 t_2}]^2 \\ &= 1/2 t_1 (t_2 - t_1) \sum_i \mathbb{E}[\varphi'(x_{s_i t_1})^2] (s_{i+1} - s_i)^2 \\ &\leq t_1 (t_2 - t_1) (s_2 - s_1) |\pi| \sup_{s \in [0, 1]^2} \mathbb{E}[\varphi'(x_{st_1})^2]. \end{aligned}$$

Then it suffices to remark that the term appearing in the r.h.s vanish when $|\pi|$ go to zero and this finishes the proof. \square

Proposition 6.10. Let $\varphi \in C^3(\mathbb{R})$ satisfying the (GC) and $\Pi = \{(s_i, t_j)\}_{ij}$ a dissection of the rectangle $[s_1, s_2] \times [t_1, t_2]$ then we have :

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} \varphi(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, xx} = 0$$

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} \varphi(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, \omega x} = 0$$

and

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_a} = 0$$

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, \omega x, b_a} = 0$$

where $J^{Rough, xx, b_a} = \int_a \varphi'(x) d_a x \int_{\hat{a}} dx dx$ and $J^{Rough, \omega x, b_a} = \int_a \varphi'(x) d_a x \int_{\hat{a}} d\omega dx$ are given by the Proposition 5.4

Proof. We will prove only the first and third statement, for the two other we have exactly the same proof. Now definition we have that $C_{s_1 s_2 t_1 t_2}^{Ito, XX} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} \hat{d}_{uv} x_{uv} \hat{d}_{st} x_{st}$ and then by independence of the increment of the Brownian sheet we have that

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i,j} \varphi(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, xx} \right|^2 \right] &= \sum_{i,j} 1/4 \mathbb{E} [\varphi(x_{s_i t_j})^2] (s_{i+1} - s_i)^2 (t_{j+1} - t_j)^2 \\ &\lesssim |\Pi| \sup_{(s,t) \in [0,1]^2} \mathbb{E} [\varphi(x_{st})^2] \end{aligned}$$

This give us the first convergence. Now we will focus on the third convergence which require more work. In fact by definition

$$\begin{aligned} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_1} &= \Lambda_1 [\delta_1 \varphi(x) C^{It\hat{o}, xx} + \varphi'(X) \mu_1 E_1^{It\hat{o}, xxx} \\ &\quad + \mu_1 (\Lambda_1 \otimes 1) (\varphi(x)^{\sharp 1} D_1^{It\hat{o}, xx} + \delta_1 \varphi'(x) E_1^{It\hat{o}, xx})]_{s_i s_{i+1} t_j t_{j+1}} \\ &= \Lambda^{1d} [\delta_1 \varphi(x)_{.t_j} C_{.t_j t_{j+1}}^{It\hat{o}, xx} + \varphi'(x)_{.t_j} (\mu_1 E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}} \\ &\quad + \mu_1 (\Lambda_1 \otimes 1) (\varphi(X)_{.t_j}^{\sharp 1} (D_1^{It\hat{o}, xx})_{.t_j t_{j+1}} + \delta_1 \varphi'(X)_{.t_j} (E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}})]_{s_i s_{i+1}} \end{aligned}$$

and then we have the bound

$$\begin{aligned} |J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_1}| &\lesssim_h (s_{i+1} - s_i)^{3h} (\|\delta_1 \varphi(X)_{.t_j}\|_h \|C_{.t_j t_{j+1}}^{It\hat{o}, xx}\|_{2h} + (\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|) \|(E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}}\|_{\mathcal{C}_2^{2h} \otimes \mathcal{C}_2^h} \\ &\quad + \|\varphi(x)_{.t_j}^{\sharp 1}\|_{2h} \|(D_1^{It\hat{o}, xx})_{.t_j t_{j+1}}\|_{\mathcal{C}^h \otimes \mathcal{C}^h} + \|\delta_1 \varphi'(x)_{.t_j}\|_h \|(E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}}\|_{\mathcal{C}^{2h} \otimes \mathcal{C}^h}) \end{aligned} \quad (90)$$

By independence of increments in the second direction, this gives:

$$\mathbb{E} \left[\left| \sum_{i,j} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_a} \right|^2 \right] \lesssim_h \left| \sum_i (s_{i+1} - s_i)^{3h} \right|^2 (a_1 + a_2 + a_3 + a_4) \quad (91)$$

where

1. $a_1 = \sum_j \mathbb{E} [\|\delta_1 \varphi(x)_{.t_j}\|_h^2] \mathbb{E} [\|C_{.t_j t_{j+1}}^{It\hat{o}, xx}\|_{2h}^2]$
2. $a_2 = \mathbb{E} [\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|^2] \sum_j \mathbb{E} [\|(E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}}\|_{\mathcal{C}_2^{2h} \otimes \mathcal{C}_2^h}^2]$
3. $a_3 = \sum_j \mathbb{E} [\|\varphi(x)_{.t_j}^{\sharp 1}\|_{2h}^2] \mathbb{E} [\|(D_1^{It\hat{o}, xx})_{.t_j t_{j+1}}\|_{\mathcal{C}^h \otimes \mathcal{C}^h}^2]$
4. $a_4 = \sum_j \mathbb{E} [\|\delta_1 \varphi'(x)_{.t_j}\|_h^2] \mathbb{E} [\|(E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}}\|_{\mathcal{C}^{2h} \otimes \mathcal{C}^h}^2]$

To obtain our convergence it suffices to show that all these terms are bounded. A simple computation gives :

$$\mathbb{E} [\|\delta_1 \varphi(x)_{.t_j}\|_h^2] \lesssim \mathbb{E} [\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|^4]^{1/2} \mathbb{E} [\|\delta_1 x\|_h^4]^{1/2}$$

where the r.h.s is finite. Now the Lemma 6.2 gives

$$\|C_{.t_j t_{j+1}}^{Ito, xx}\|_{2h}^p \lesssim_{h,p} \int_{[0,1]^2} \frac{|C_{u_1 u_2 t_j t_{j+1}}^{It\hat{o}, xx}|^p}{|u_2 - u_1|^{2hp+2}} du_1 du_2 + \int_{[0,1]^4} \frac{|(D_1^{It\hat{o}, xx})_{u_1 u_2 u_3 u_4 t_j t_{j+1}}|^p}{|u_4 - u_3|^{hp+2} |u_2 - u_1|^{hp+2}} du_1 du_2 du_3 du_4$$

then taking the expectation in this last equality and using Jensen inequality we obtain that

$$\mathbb{E}[\|C_{.t_j t_{j+1}}^{Ito, XX}\|_{2h}^2] \lesssim_h c_p (t_{j+1} - t_j)^2 < \infty$$

where $c_p < +\infty$ for p large enough and then

$$a_1 \lesssim_{h,p} \mathbb{E}[\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|^4]^{1/2} \mathbb{E}[\|\delta_1 x\|_h^4]^{1/2} \sum_j (t_{j+1} - t_j)^2$$

the terms a_2, a_3 and a_4 can be treated similarly and then we obtain the wanted convergence of the boundary integral . \square

Corollary 6.11. *Let $\varphi \in C^5(\mathbb{R})$ satisfying the (GC) then the integral*

$$\mathcal{J}_{s_1 s_2 t_1 t_2}^{It\hat{o}} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) \hat{d}_{st} x_{st}$$

where \hat{d} is a Ito differential admit a continuous version with coincide with

$$\mathcal{J}_{s_1 s_2 t_1 t_2}^{Rough} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) d_{st} x_{st}$$

where this last integral are given by Theorem 5.6

Proof. let $\Pi = \{(s_i, t_j)\}_{ij}$ a dissection of the rectangle $[s_1, s_2] \times [t_1, t_2]$ then by definition we have that

$$\begin{aligned} \mathcal{J}_{s_1 s_2 t_1 t_2}^{Rough} &= \sum_{ij} (-\varphi(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} + \varphi'(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, xx} + \varphi'(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, \omega x} \\ &\quad + \sum_{a=1,2} (I_{s_i s_{i+1} t_j t_{j+1}}^{Rough, x, b_a} + J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_a} + J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, \omega x, b_a}) + r_{s_i s_{i+1} t_j t_{j+1}}^b) \end{aligned} \quad (92)$$

where $r^b \in \mathcal{C}_{2,2}^{1+,1+}$ this fact combined with the proposition (6.10) and (6.9) give us

$$\mathcal{J}_{s_1 s_2 t_1 t_2}^{Rough} = \mathbb{P} - \lim_{|\Pi| \rightarrow 0} \sum_{ij} -\varphi(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} + I_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, x, b_1} + I_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, x, b_2}$$

Now this last converge to the Ito integral in fact

$$\mathbb{E}[\|\mathcal{J}_{s_1 s_2 t_1 t_2}^{It\hat{o}} - \sum_{ij} I_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, x, b_1}\|^2] \lesssim \mathbb{E}[\sup_{(s,t) \in [0,1]^2} |\varphi(x_{st})|^4]^{1/2} \mathbb{E}[\sup_{|t-t'|+|s-s'| \leq |\Pi|} |x_{st} - x_{s't'}|^4]^{1/2}$$

which the r.h.s vanish when the mesh of the partition go to zero then we have the result. \square

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